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THE COMPUTER IMPROVES THE STEINER'S CONSTRUCTION OF THE MALFATTI CIRCLES

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Abstract. The computer program "Discoverer", created by the authors, is the first computer program, which is able easily to discover new theorems in Mathematics, and possibly, the first computer program,which is able easily to discover new knowledge in science. In this paper we give a detailed description of an improvement of the classical Steiner's solution of the construction of the Malfatti circles, discovered by the computer program "Discoverer". We use the theory of the complexity of the geometric constructions in order to obtain a numerical measure of the complexity of the solutions.

Keywords: computer-generated mathematics, Euclidean geometry, Discoverer, Malfatti circles, Steiner's solution.

1. Introduction

The computer program "Discoverer" is the first computer program, which is able easily to discover new theorems in mathematics, and possibly, the first computer program, which is able easily to discover new knowledge in science, see (Grozdev & Dekov, 2013, 2014a,b,c).

In this paper we give a detailed description of an improvement of the Steiner's solution of the Malfatti circles construction, created by the "Discoverer" (Grozdev & Dekov, 2013). We use the theory of the complexity of the geometric constructions in order to obtain a numerical measure of the complexity of the solutions.

The construction of the Malfatti circles is one of the famous mathematical problems. The problem was posed by the Italian geometer Gian Francesco Malfatti in 1803. A simple construction of the Malfatti circles with a compass and a ruler has been published by the great Swiss geometer Jacob Steiner in 1826[1] (Tabov & Lazarov, 1990). As far as the authors know, the improvement of the Steiner's construction of the Malfatti circles, discovered by the"Discoverer", is the first essential improvement of an important result in Mathematics, discovered by a computer, and possibly, the first improvement of an important result in science, discovered by a computer.

2. Complexity of geometric constructions

The first measure of the complexity of geometric constructions is proposed by Lemoine[1] (Tabov & Lazarov, 1990). In this paper we use the measure of Lazarov and Tabov

(Tabov & Lazarov, 1990) which is summarized in Table 1. This measure specifies the Lemoine's measure. The explanation of row 1 in Table 1 is as follows. To place the edge of the ruler in coincidence with a point (Lemoine's operation R_1) – one point. To place the edge of the ruler in coincidence with a second point – one point. To draw a straight line (Lemoine's operation R_2) – one point. Hence, we obtain 3 points for drawing a straight line. The explanation of rows 2 and 3 in Table 1 is similar.

	Construction	**Lazarov-Tabov complexity**
1	Construct a line, which passes through two points.	3
2	Construct a circle with a given center and passing through another point.	3
3	Construct a circle with a given center and a radius, given by two points which are different form the center.	4
4	Construct a point, which is the intersection of two lines, circles, or a line and a circle.	1
5	Construct a point, which lies on a geometric figure or outside a geometric figure.	1

Table 1

We use aslo the Fransois Labelle's measure[2]. The Labelle's measure is as follows: the complexity of a construction is defined to be the number of drawing operations (lines and circles) that are performed.

In the header cells in the tables below "LT" means "Lazarov-Tabov complexity" and "L" means "Labelle's complexity".

We illustrate the measures with four examples. We will use these examples in the next sections.

Example 1. See Figure 1. Construct an internal angle bisector of a given angle. The edge of the angle is labeled A, and the rays of the angle are labeled R_1 and R_2. The construction and the complexity are given in Table 2.

Step	**Construction**	**LT**	**L**
1	B = point constructed on ray R_1.	1	0
2	c = circle with center point A through point B.	3	1
3	C = intersection point of ray R_2 and circle c.	1	0
4	c_1 = circle with center point B through point C.	3	1

5	c_2= circle with center point *C* through point *B*.	3	1
6	*D* = intersection point of circles c_1 and c_2.	1	0
7	*L* = ray with endpoint *A* and passing through point *D* = internal bisector of the given angle.	3	1
Total complexity		15	4

Table 2

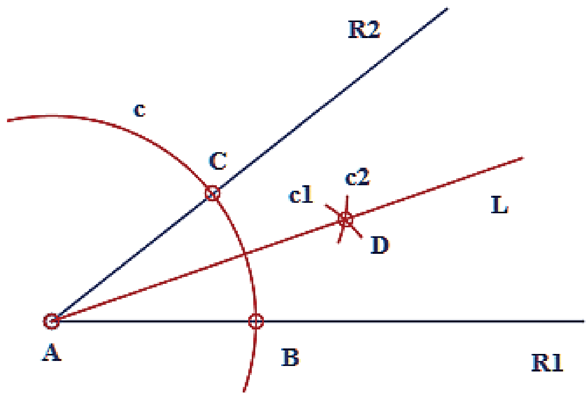

Figure 1

Example 2. See Figure 2. Construct the projection of a point *A* on a line *L*. See Table 3.

Step	Construction	LT	L
1	*B* = point constructed on line *L*.	1	0
2	*c* = circle with center point *A* through point *B*.	3	1
3	*C* = intersection point of line *L* and circle *c*.	1	0
4	c_1 = circle with center point *B* through point *C*.	3	1
5	c_2 = circle with center point *C* through point *B*.	3	1
6	*D* = intersection point of circles c_1 and c_2.	1	0
7	L_1= line through points *A* and *D*.	3	1
8	*M* = intersection point of lines *L* and L_1 = projection of point *A* on line *L*.	1	0
Total complexity		16	4

Table 3

Example 3. See Figure 3. Construct the reflection of a point A in a line L. See Table 4.

Step	Construction	LT	L
1	B = point constructed on line L	1	0
2	c = circle with center point A through point B.	3	1
3	C = intersection point of line L and circle c.	1	0
4	c_1= circle with center point B through point A.	3	1
5	c_2 = circle with center point C through point A.	3	1
6	D = intersection point of circles c_1 and c_2, different from point A = reflection of point A in line L.	1	0
	Total complexity	12	3

Table 4

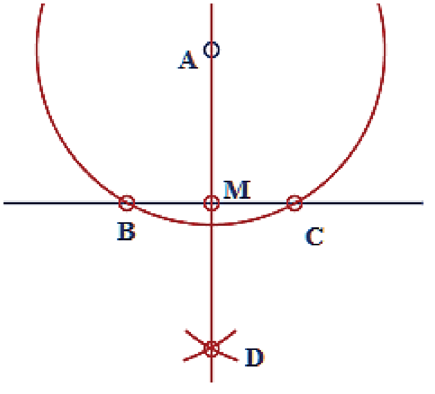

Figure 2

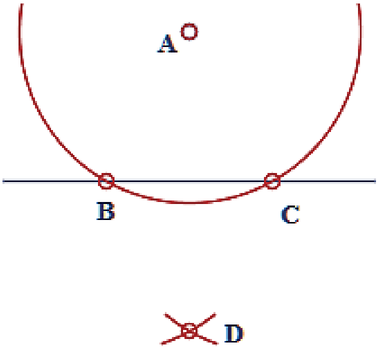

Figure 3

Example 4. See Figure 4. Given circle c, line L tangent to circle c and a second line L_1. Construct the tangent to c, different from L, through the point which is the intersection of the lines L and L_1. See Table 5. Below " L_2 = second tangent (c,L, L_1)" means that L_2 is the second tangent to c, constructed as in Table 5. We use this notation below in Table 6, rows 18 and 19.

Step	Construction	LT	L
1	A = intersection point of lines L and L_1.	1	0
2	B = intersection point of line L and circle c.	1	0
3	c_1= circle with center point A through point B.	3	1

4	C = intersection point of circles c and c_1, different from point B.	1	0
5	L_2= line trough points A and C = second tangent line from point A to circle c.	3	1
	Total complexity	9	2

Table 5

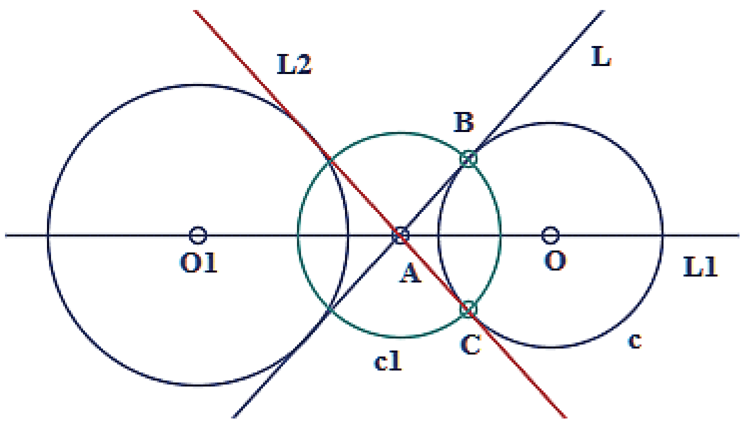

Figure 4

3. Steiner's solution

The Malfatti problem is as follows: Within a given triangle draw three circles each of which is tangent to the other two and to two sides of the triangle.

Given ΔABC, that is, given points A, B and C and lines BC, CA and AB. The Steiner's construction has the following stages:

Stage 1. Construct the internal angle bisectors and the incenter of ΔABC.

Stage 2. Construct the vertices of the de Villiers triangle.

Stage 3. Construct the Malfatti-Steiner point.

Stage 4. Construct the Malfatti central triangle.

Stage 5. Construct the Malfatti circles.

The steps of the construction are given in Table 6. In Table 6, the internal angle bisectors, labeled by L_1, L_2 and L_3, are constructed at steps 1, 2 and 3, and the incenter, labeled by I, is constructed at step 4. The vertices of the de Villiers triangle, labeled by V_1, V_2 and V_3, are constructed at steps 7, 10 and 15. The Malfatti-Steiner point, labeled by S, is constructed at step 20. The vertices of the Malfatti central triangle, labeled by O_1, O_2 and O_3, are constructed at steps 24, 26 and 28. The Malfatti circles, labeled by c_1, c_2 and c_3, are constructed at steps 30, 32 and 34. See Figures 5-9.

Stage	Step	Construction	LT	L
1	1	L_1 = internal bisector of $\angle BAC$.	15	4
	2	L_2= internal bisector of $\angle CBA$.	15	4
	3	L_3= internal bisector of $\angle ACB$.	15	4
	4	I = intersection of lines L_1 and L_2.	1	0
2	5	L_{v11}= internal bisector of $\angle CBI$.	15	4
	6	L_{v12} = internal bisector of $\angle BCI$.	15	4
	7	V_1= intersection of lines L_{v11} and L_{v12}.	1	0
	8	L_{v21} = internal bisector of $\angle ACI$.	15	4
	9	L_{v22} = internal bisector of $\angle CAI$.	15	4
	10	V_2 = intersection of lines L_{v21} and L_{v22}.	1	0
	11	L_{v31} = internal bisector of $\angle BAI$.	15	4
	12	L_{v32} = internal bisector of $\angle ABI$.	15	4
	13	V_3 = intersection of lines L_{v31} and L_{v32}.	1	0
3	14	L_{13}= line through points V_1 and V_3.	3	1
	15	L_{23}= line through points V_2 and V_3.	3	1
	16	Z = projection of point V_3 on line AB.	16	4
	17	c_{v3} = circle with center V_3 through Z.	3	1
	18	L_{S1} = second tangent (c_{v3}, L_1, L_{23}).	9	2
	19	L_{S2} = second tangent (c_{v3}, L_2, L_{13}).	9	2
	20	S = intersection of lines L_{S1} and L_{S2}.	1	0
4	21	X = intersection point of lines BC and L_{S1}.	1	0
	22	Y = intersection point of lines CA and L_{S2}.	1	0
	23	L_{c1} = internal bisector of $\angle AYS$.	15	4
	24	O_1 = intersection point of lines L_1 and L_{c1}.	1	0

	25	L_{c2} = internal bisector of $\angle BXS$.	15	4
	26	O_2 = intersection point of lines L_2 and L_{c2}.	1	0
	27	L_{c3} = internal bisector of $\angle CXS$.	15	4
	28	O_3 = intersection point of lines L_3 and L_{c3}.	1	0
5	29	X_1 = projection of point O_1 on line CA.	16	4
	30	c_1 = circle with center O_1 through X_1.	3	1
	31	X_2 = projection of point O_2 on line AB.	16	4
	32	c_2 = circle with center O_2 through X_2.	3	1
	33	X_3 = projection of point O_3 on line BC.	16	4
	34	c_3 = circle with center O_3 through X_3.	3	1
		Total complexity	290	74

Table 6

We see that the complexity of the Steiner's solution is 290, if we use the Lazarov-Tabov measure, and the complexity is 74, if we use the Labelle's measure.

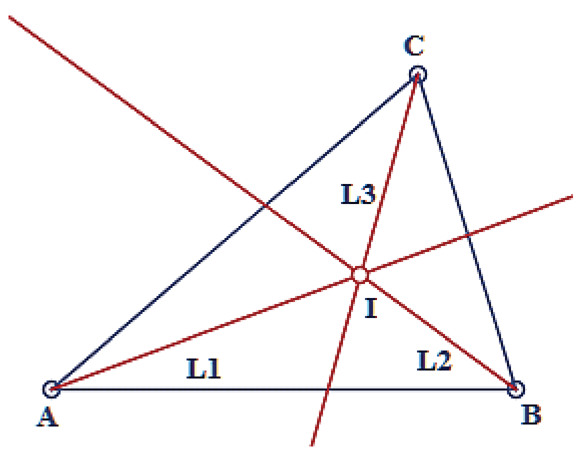

Figure 5. Stage 1 of the Steiner's construction.

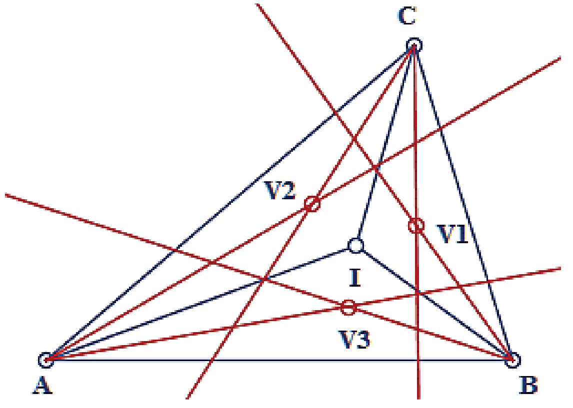

Figure 6. Stage 2 of the Steiner's construction.

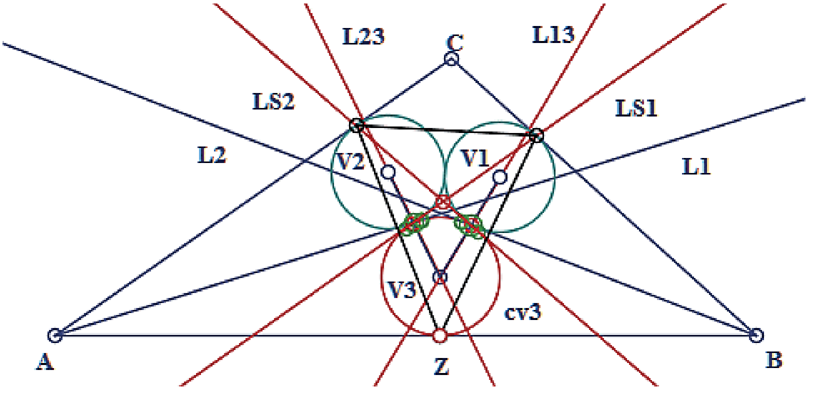

Figure 7. Stage 3 of the Steiner's construction.

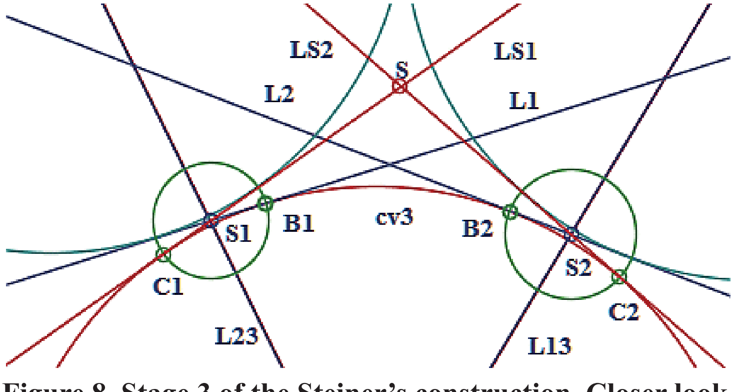

Figure 8. Stage 3 of the Steiner's construction. Closer look.

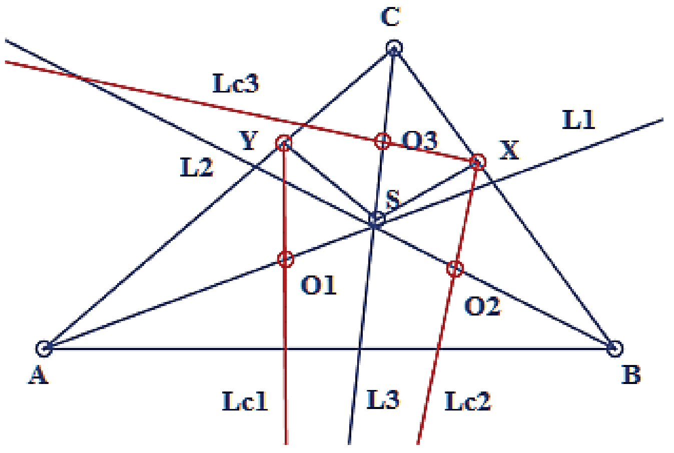

Figure 9. Stage 4 of the Steiner's construction.

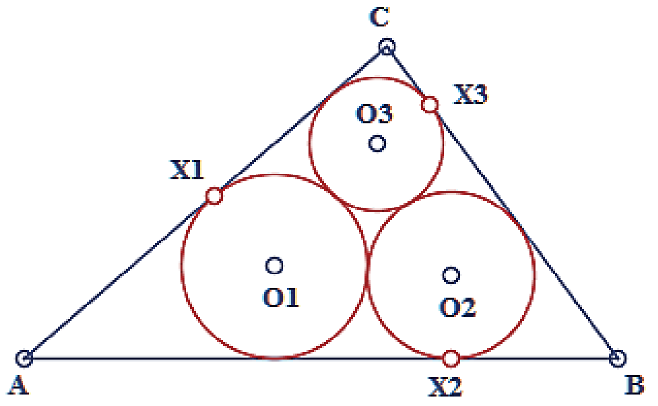

Figure 10. Stage 5 of the Steiner's contruction.

4. Replacements

A replacement of stage 3 of the Steiner's solution is given by Richard Guy[3] (Guy, 2007). The Guy's construction of stage 3 is given in Table 7. See Figure 11.

Step	Construction	LT	L
1	L_{13} = line through points V_1 and V_3.	3	1
2	L_{23}= line through points V_2 and V_3.	3	1
3	M_1 = intersection point of lines L_1 and L_{23}.	1	0
4	A_S = reflection of point A in line L_{23}.	12	3

5	L_{S1} = line through points M_1 and A_S.	3	1
6	M_2 = intersection point of lines L_2 and L_{13}.	1	0
7	B_S = reflection of point B in line L_{13}.	12	3
8	L_{S2} = line through points M_2 and B_S.	3	1
9	S = intersection point of lines L_{S1} and L_{S2}.	1	0
	Total complexity	39	10

Table 7

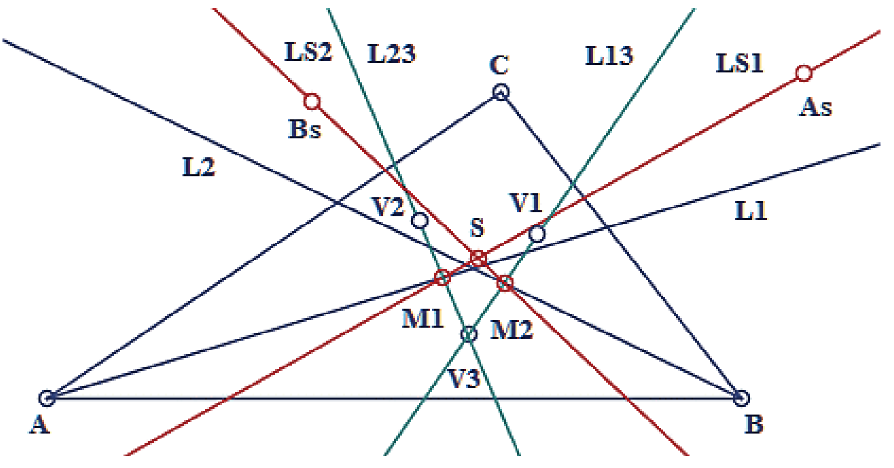

Figure 11. The Guy's construction of stage 3

A replacement of stage 4 of the Steiner's solution is given by the computer program "Discoverer" (Grozdev & Dekov, 2013). The construction of "Discoverer" of stage 4 is given in Table 8. See Figure 12.

Step	Construction	LT	L
1	L_{o1} = line through points S and V_1.	3	1
2	O_1 = intersection point of lines L_1 and L_{o1}.	1	0
3	L_{o2} = line through points S and V_2.	3	1
4	O_2 = intersection point of lines L_2 and L_{o2}.	1	0
5	L_{o3} = line through points S and V_3.	3	1
6	O_3 = intersection point of lines L_3 and L_{o3}.	1	0
	Total complexity	12	3

Table 8

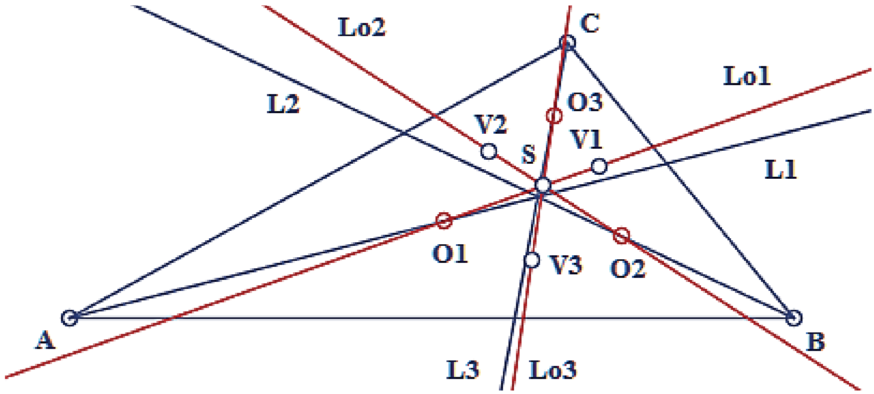

Figure 12. The Discoverer's construction of stage 4

Table 9 gives comparison of the variants of the Steiner's construction.

	Construction	LT	L
1	Steiner's construction.	290	74
2	Steiner's construction where stage 3 is replaced by the Guy's construction of stage 3.	285	73
4	Steiner's construction where stage 4 is replaced by the Discoverer's construction of stage 4.	252	65
5	Steiner's construction where stage 3 is replaced by the Guy's construction of stage 3, and stage 4 is replaced by the Discoverer's construction of stage 4.	247	64

Table 9

5. Conclusion

The Lazarov-Tabov measure of stage 4 of the Steiner's construction is 50, while the improved by "Discoverer" stage 4 has measure 12. The Labelle's measure of stage 4 of Steiner's construction is 12, while the improved by "Discoverer" stage 4 has measure 3. The complexity of the improved by "Discoverer" stage 4 is in the first case 24%, and in the second case 25% of the complexity of the Steiner's stage 4. Hence, the computer program "Discoverer" has discovered an essential improvement of stage 4 of the Steiner's construction of the Malfatti circles.

NOTES

1. Geometrography, http://en.wikipedia.org/wiki/
2. Labelle, F., The complexity of geometric constructions, http://www.cs.mcgill.ca/~sqrt/cons/constructions.html

3. Malfatti circles, http://en.wikipedia.org/wiki/

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КОМПЮТЪРЪТ ПОДОБРЯВА КОНСТРУКЦИЯТА НА ЩАЙНЕР НА ОКРЪЖНОСТИТЕ НА МАЛФАТИ

Резюме. Компютърната програма "Откривател", създадена от авторите, е първата компютърна програма, която може лесно да открива нови теореми в математиката, и може би първата компютърна програма, която може да прави открития в науката изобщо. В тази статия предлагаме подробно описание на едно опростяване на класическото решение на Щайнер за построяване на окръжностите на Малфати, открито от компютърната програма "Откривател". Използваме теорията на сложността на геометричните построения, за да получим числена мярка на сложността на решенията.

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