

ONE MORE PROOF FOR THE DISTANCE BETWEEN THE INCENTRE AND THE ORTHOCENTRE OF THE TRIANGLE

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Abstract. One more proof is proposed in the present paper concerning the the distance between the incentre (I) and the orthocentre (H) of the triangle.

Keywords: distance, incentre, orthocentre, sine law, cosine law, centroid of a triangle.

The following equality is considered in the paper:

$$|IH|^2 = 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma, \quad (1)$$

where r and R are respectively the radii of the incircle and the circumcircle of the triangle $\triangle ABC$ with angles α , β and γ . One proof of the equality, which is not simple at all, is proposed by the present author in his paper (Arslanagić, 2005, pp. 437-439). For a new proof the next two equalities will be used:

$$|IO|^2 = R^2 - 2Rr, \quad (2)$$

where O is the circumcentre of the triangle and

$$|OH|^2 = 9r^2 - (a^2 + b^2 + c^2), \quad (3)$$

where a , b , c are the lengths of its sides.

Two proofs of the equality (2) are proposed by the present author in (Arslanagić, 2005, pp. 432-434) and one my proof in (Arslanagić, 2009, pp. 79-80). Similarly, two proofs of the equality (3) are proposed in (Arslanagić, 2005, pp. 435-437).

One more equality will be used in the sequel:

$$|IT|^2 = \frac{2}{3}r^2 - \frac{4}{3}Rr + \frac{1}{18}(a^2 + b^2 + c^2), \quad (4)$$

where T is the centroid of the triangle ΔABC . Here is a proof of (4):

Proof: The idea is to apply Leibniz theorem for the triangle ΔABC , see (Arslanagić, 2005, pp. 354-355) or (Grozdev, 2007), i.e.:

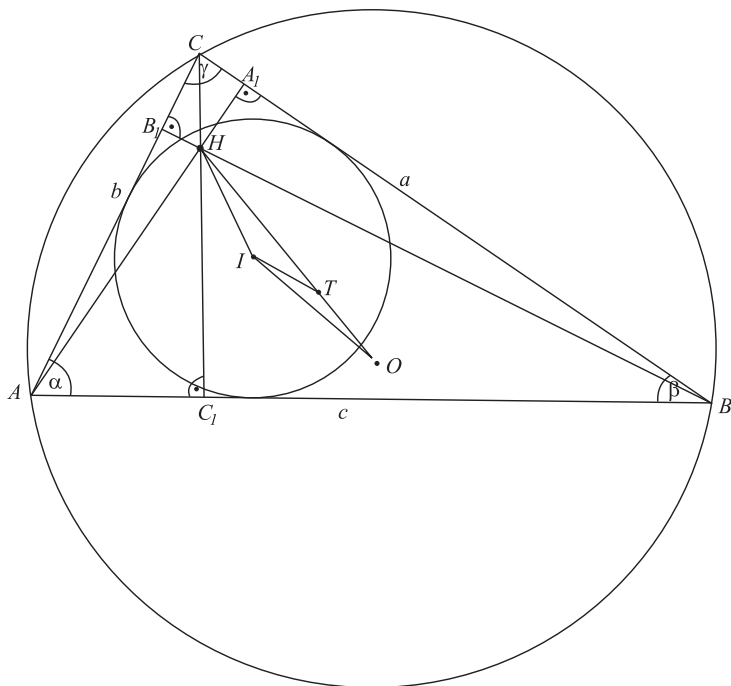
$$|MA|^2 + |MB|^2 + |MC|^2 = 3|MT|^2 + |AT|^2 + |BT|^2 + |CT|^2, \quad (5)$$

where the point M is not necessary in the plane of the triangle ΔABC . Since $|AT| = \frac{2}{3}m_a$, $|BT| = \frac{2}{3}m_b$, $|CT| = \frac{2}{3}m_c$, (m_a, m_b, m_c are the medians of the triangle ΔABC), we have:

$$|AT|^2 + |BT|^2 + |CT|^2 = \frac{4}{9}(m_a^2 + m_b^2 + m_c^2) = \frac{4}{9} \cdot \frac{3}{4}(a^2 + b^2 + c^2) = \frac{1}{3}(a^2 + b^2 + c^2).$$

By the substitutions $|MA| = x$, $|MB| = y$, $|MC| = z$ we get:

$$9|MT|^2 = 3(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2).$$



If M is replaced by I , then:

$$\begin{aligned}
 x^2 &= (s-a)^2 + r^2, \quad y^2 = (s-b)^2 + r^2, \quad z^2 = (s-c)^2 + r^2; \quad \left(s = \frac{1}{2}(a+b+c) \right) \\
 9|IT|^2 &= 3 \left[(s-a)^2 + r^2 + (s-b)^2 + r^2 + (s-c)^2 + r^2 \right] - (a^2 + b^2 + c^2) \\
 \Rightarrow 9|IT|^2 &= 3 \left[(3s^2 + 2s(a+b+c) + a^2 + b^2 + c^2 + 3r^2) \right] - (a^2 + b^2 + c^2) \\
 &\Rightarrow 9|IT|^2 = 9r^2 - 3s^2 + 2(a^2 + b^2 + c^2). \tag{6}
 \end{aligned}$$

Since $a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8Rr$, then.

$$\begin{aligned}
 2s^2 &= a^2 + b^2 + c^2 + 2r^2 + 8Rr \\
 \Rightarrow s^2 &= \frac{1}{2}(a^2 + b^2 + c^2) + r^2 + 4Rr. \tag{7}
 \end{aligned}$$

From (6) and (7) we get:

$$\begin{aligned}
 9|IT|^2 &= 9r^2 - \frac{3}{2}(a^2 + b^2 + c^2) - 3r^2 - 12Rr + 2(a^2 + b^2 + c^2) \\
 \Rightarrow |IT|^2 &= \frac{2}{3}r^2 - \frac{4}{3}Rr + \frac{1}{18}(a^2 + b^2 + c^2),
 \end{aligned}$$

and this is the equality (4).

Further, we give the proof of the equality (1).

By the cosine law for the triangle ΔTIO :

$$|IT|^2 = |IO|^2 + |OT|^2 - 2|IO| \cdot |OT| \cos \varphi, \tag{8}$$

where $\varphi = \angle IOT$.

It is well known that the points O , T and H are collinear, lying on the **Euler's line**.

We will use the corresponding **Euler's theorem**:

$$|OT| = \frac{1}{3}|OH|. \tag{9}$$

From (8) and (9) it follows, that

$$\cos \varphi = \frac{|IO|^2 + |OT|^2 - |IT|^2}{2|IO| \cdot |OT|}$$

$$\begin{aligned}\Rightarrow \cos \varphi &= \frac{|IO|^2 + \frac{1}{9}|OH|^2 - |IT|^2}{2|IO| \cdot \frac{1}{3}|OH|} \\ \Rightarrow \cos \varphi &= \frac{9|IO|^2 + |OH|^2 - 9|IT|^2}{6|IO| \cdot |OH|}.\end{aligned}\quad (10)$$

Apply now the cosine law for the triangle ΔIHO :

$$|IH|^2 = |IO|^2 + |OH|^2 - 2|IO| \cdot |OH| \cos \varphi,$$

and from here, because of (10), we have:

$$\begin{aligned}|IH|^2 &= |IO|^2 + |OH|^2 - 2|IO| \cdot |OH| \cdot \frac{9|IO|^2 + |OH|^2 - 9|IT|^2}{6|IO| \cdot |OH|} \\ \Rightarrow |IH|^2 &= |IO|^2 + |OH|^2 - 3|IO|^2 - \frac{1}{3}|OH|^2 + 3|IT|^2 \\ \Rightarrow |IH|^2 &= -2|IO|^2 + \frac{2}{3}|OH|^2 + 3|IT|^2.\end{aligned}\quad (11)$$

Finally, from (11) on the base of (2), (3) and (4), we obtain:

$$\begin{aligned}|IH|^2 &= -2R^2 + 4Rr + 6R^2 - \frac{2}{3}(a^2 + b^2 + c^2) + 2r^2 - 4Rr + \frac{1}{6}(a^2 + b^2 + c^2) \\ \Rightarrow |IH|^2 &= 4R^2 + 2r^2 - \frac{1}{2}(a^2 + b^2 + c^2),\end{aligned}$$

next by the sine law:

$$\Rightarrow |IH|^2 = 4R^2 + 2r^2 - \frac{1}{2} \cdot 4R^2 (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma),$$

and by the well known identity

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2 + 2 \cos \alpha \cos \beta \cos \gamma$$

we have:

$$\begin{aligned}|IH|^2 &= 4R^2 + 2r^2 - 2R^2 (2 + 2 \cos \alpha \cos \beta \cos \gamma), \text{ i.e.} \\ |IH|^2 &= 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma.\end{aligned}$$

Corollary. Prove the inequality 2.23 from (Bottema & al., 1969, p. 25), i.e.

$$\cos \alpha \cos \beta \cos \gamma \leq \frac{1}{8}, \quad (12)$$

where the equality holds true if and only if $\alpha = \beta = \gamma$, i.e. if the triangle is equilateral.

Proof. Since $|IH|^2 \geq 0$, it follows from (1), that:

$$\begin{aligned} 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma &\geq 0 \\ \Rightarrow \cos \alpha \cos \beta \cos \gamma &\leq \frac{r^2}{2R^2}. \end{aligned} \quad (13)$$

It is well known the inequality $R \geq 2r$ (it follows from (2): $|IO|^2 \geq 0 \Rightarrow R^2 - 2Rr \geq 0 \Rightarrow R \geq 2r$). Thus, $\frac{r}{R} \leq \frac{1}{2} \Rightarrow \frac{r^2}{R^2} \leq \frac{1}{4} \Rightarrow \frac{r^2}{2R^2} \leq \frac{1}{8}$ and we conclude that the inequality (13) is stronger than the inequality (12). The equality in (13) holds true for the equilateral triangle too.

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ТРИ РЕШЕНИЯ НА ЕДНА ЗАДАЧА С ЧЕТИРИ ОКРЪЖНОСТИ

Резюме. Статията е посветена на три решения на една интересна задача за четири окръжности.

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