Educational Research
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STATIONARY NUMBERS

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Abstract. The article studies one method of numbers generation. For this new method we define and study sets of so called *m*-digitaddition and *m*-self positive integers. In addition, we introduce a stationary number term for the mentioned operation and provide a full description of the set of stationary numbers under some conditions.

Keywords: Kaprekar, generator, digitaddition, self number.

1. Introduction. In his book "Time Travel and Other Mathematical Bewilderments" (Gardner, 1988) the famous American science writer Martin Gardner writes about one Indian mathematician D. R. Kaprekar, who has discovered a remarkable set of so called digitaddition numbers. Let us choose any positive integer *n* and denote the sum of its digits by $S(n)$. The number $K(n)=n+S(n)$ is called a *digitaddition* and the chosen number *n* is its *generator*. For example, if we choose the number 53, then its digitaddition is $53+5+3=61$.

A digitaddition may have more than one generator. The least digitaddition with two generators is 101, it is generated by 91 and 100. The least digitaddition with tree generators, $10^{13}+1$, is generated by $10^{13}, 10^{13}-99, 10^{13}-108$. The least digitaddition with four generators discovered by Kaprekar, $10^{24}+102$, has 25 digits. He managed to find the least digitadditions with 5 and 6 generators as well.

A positive integer that has no generator is called a *self number*. An article in the American journal "The American Mathematical Monthly" of 1974 showed that there exist infinitely many self numbers, but they are far less frequent than digitadditions. There are only 13 self numbers in the first hundred: 1, 3, 5, 7, 9, 20, 31, 42, 53, 64, 75, 86, 97. A million, i.e. 10^6, is a self number and the next power of ten self number is 10^{16}. There are self numbers like 11 111 111 111 111 111 111 and 3 333 333 333 as well. Non-recursive formula for self numbers is yet to be discovered.

$K(n)$, basically, is a new number, generated by *n* with the use of simple and natural function. This article proposes another natural and rather simple procedure of generating new numbers. In terms of this new operation classes of *m*-digitaddition and *m*-self numbers are then defined, some facts about those two classes are found. In addition, a term of stationary number is defined and, with some conditions, the set of such numbers is fully described.

2. Definitions. Let $I=\{0, 1, \dots, 9\}$ be the set of the decimal digits and let N be the set of the positive integers. If $a \in N$, then a can be expressed as $a=\alpha_{k-1}\cdot 10^{k-1}+\alpha_{k-2}\cdot 10^{k-2}+\ldots+\alpha_1\cdot 10+\alpha_0$ where $\alpha_{k-1}\neq 0$ and $\alpha_i \in I$ $(i=0, 1,..., k-1)$. We will denote a as $a=(\alpha_{k-1},\alpha_{k-2},\dots,\alpha_0)$ and call number $k=d(a)$ its *rank*, or simply the number of digits. By definition, $10^{k-1}\leq a \leq 10^k-1$.

Let $S(a)=\alpha_{k-1}+...+\alpha_0$ be the sum of a's digits. The number

$$\hat{a}=(\alpha_0\ ...\ \alpha_{k-1})=\alpha_0 10^{k-1}+\alpha_1 10^{k-2}+...+\alpha_{k-1}$$

will be called *backward* to a. Some of $\hat{a}$'s first digits can be zeros, thus $d(\hat{a})\leq k$. If $a=\hat{a}$, then a is called *symmetrical*.

Kaprekar was studying the sum of a number and its digits: $a+s(a)$. If we add $\hat{a}$ to that expression it becomes symmetrical: $a+s(a)+\hat{a}$. That expression is greater than a and always devisable by 3, thus it seems logical to consider only a third part of it:

$$M(a)=\frac{1}{3}\left(a+s(a)+\hat{a}\right).$$

We have just built a quite natural and simple procedure for generating new numbers: $a \to M(a)$. Following the example of Kaprekar, $M(a)$ will be called an *m*-digitaddition with an *m*-generator a. Numbers without *m*-generators will be called *m*-self.

If we denote the set of all *m*-self numbers by *E* and the set of all *m*-digitadditions by *G*, then

$$N=G\cup E.$$

Let's deduce some properties of the operation $a \to M(a)$.

a) If $a=(\alpha_{k-1}...\alpha_0)$ and $\alpha_0\neq 0$, then

$$M(a)=M(\hat{a}).$$

б) If $d(a)=k$, then $k-1\leq d(M(a))\leq k$. Because of that all the numbers in a sequence $a_1=a,\ a_2=M(a_1), a_3=M(a_2), ...$ do not exceed 10^k. It implies that an infinite sequence $a_1, a_2, ...$ will start repeating with the period of some length $l\geq 1$.

в) There are different possible relations between a and $M(a)$, for example

$$a_1<M(a_1), \text{ if } a_1\in\{19,\ 109,\ 1009,\ ...\},$$
$$a_2=M(a_2), \text{ if } a_2\in\{12,\ 102,\ 1002,\ ...\},$$
$$a_3>M(a_3), \text{ if } a_3\in\{90,\ 900,\ 9000,\ ...\}.$$

3. *m*-digitadditions. We have already said that digitadditions can be found more frequently than self numbers. In our case the situation is completely different. Thus, among the first thousand there are 773 *m*self numbers and 227 *m*-digitadditions. Among the second thousand there are 944 *m*self numbers and only

56 *m*digitadditions. Using a simple $\tilde{N}++$ code all the *m*-digitadditions in range from 1 to 10^6 were found. Their number turned out to be 15840.

Let's denote by g_r the least *m*-digitaddition that has exactly *r* *m*-generators. From the data generated by a computer program we created the three following tables.

r	1	2	3	4	5	6	7	8	9
g_r	1	4	8	16	20	24	28	32	36

We can see that $g_{i+1} - g_i$ for $i = 5, 6, 7, 8$.

r	10	20	30	40	50	60	70	80	90
g_r	334	1001	1335	1669	2003	2337	2671	3005	3339

In this table $g_{j+10} - g_j = 334$ for $j = 20,\ 30,\ 40,\ 50,\ 60,\ 70,\ 80$.

r	100	200	300	400	500	600	700	800	900
g_r	66670	100004	133338	166672	200006	233340	266674	300008	333342

Here we have $g_{l+100} - g_l = 33334$ for $l = 100, 200, 300, 400, 500, 600, 700, 800$.

The largest 6-digit *m*-digitaddition is 999973 with 18 *m*-generators.

8 has the largest amount of 3 *m*-generators among 1-digit numbers.

36 and 40 have the largest amount of 9 *m*-generators among 2-digit numbers.

964 has the largest number of *m*-generators among 3-digit numbers. It has 18 of them.

Among 4-digit numbers 3339 and 3673 have maximum number of *m*-generators: 90.

96667 and 99637 have 180 *m*-generators, which is the largest number for 5-digits.

Finally, 6-digit numbers have not more than 900 *m*-generators. Two numbers that have exactly that amount are 333342 and 366676.

4. *m*-self numbers. The following facts were found by studying all the *m*-self numbers from 1 to 10^6:

а) numbers in the form 10^p for $p = 1,\ 2,\ 3,\ 4,\ 5,\ 6$ are *m*-self;

б) numbers written with the same digit, save 5555, are *m*-self;

in particular, 11, 111, 1111, 11111, 111111, 33, 333, 3333, 33333, 333333, 99, 999, 9999, 99999, 999999 are *m*self;

в)numbersinforms $(\alpha 000)$, $(\beta 0000)$, $(\gamma 00000)$ for $\alpha, \beta, \gamma \in \{2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$ are *m*self.

The amount of *m*self numbers among the first million is 984160.

Now, we will add some definitions and notations. For any number $\alpha \in (\alpha_{k-1}...\alpha_0)$, $\alpha_{k-1} \neq 0$ digits a_{k-1-i} and a_i $(i = 0, 1, 2, ...)$ are called *symmetrical* and a sequence $\{ a_{k-1-i},\ a_i \}$ is called a pair of symmetrical digits. When we switch two symmetrical digits with their places it will be called a *symmetrical change*. Applying symmetrical change to all the symmetrical pairs of *a* we get a set P_a of numbers *similar* to *a*. If $b \in P_a$, then $P_b = P_a$. Thus, all *k*-ranked numbers are sliced into classes of similar numbers: $N_k = P_a \cup P_b \cup P_c ...$. For symmetrical *a* $|P_a| = 1$, for non-symmetrical *a* $|P_a| \geq 2$. Additionally, we can notice that every number in P_a is an *m*-generator of the same number $M(a)$.

For every $k \in N$ we denote by N_k, G_k, E_k the set of all the *k*-digit numbers, the set of *k*-digit *m*digitadditions and the set of *k*-digit *m*-selves, respectively. Clearly we have $N_k = G_k \cup E_k$. For every subset *A* in N $|A|$ denotes the number of the elements in *A*. We know that $|N_k| = 9 \cdot 10^{k-1}$.

Theorem 1. The set *E* of *m*-self numbers is infinite.

Proof. It is sufficient to show that for every $k \geq 6$ $|E_k| > 0$. The fraction of symmetrical numbers in N_k is quickly diminishing when k is growing, thus, we won't consider them to simplify the proof.

Let $A = \{ a \in N \mid d(M(a)) = k \}$ be the set of all the *m*generators of G_k. Obviously, $A = A_k \cup A_{k+1}$, where $A_k \subset N_k$ and $A_{k+1} \subset N_{k+1}$. We can see that $|A_k| < |N_k| = 9 \cdot 10^{k-1}$. If $b \in A_{k+1}$ and $b = \beta_k 10^k + \beta_{k-1} 10^{k-1} + ... + \beta_0$, $\beta_k \neq 0$, then $d(M(b)) = k$, and only 3 outcomes are possible:

$\{\beta_k = 2; \beta_0 = 0\}$, $\{\beta_k = 1; \beta_0 = 0\}$, $\{\beta_k = 1; \beta_0 = 1\}$. Thus, $|A_{k+1}| < 3 \cdot 10^{k-1}$.

Let $G_k = B_k \cup B_{k+1}$, where $B_k = \{ M(a) \mid a \in A_k \}$. $B_{k+1} = \{ M(a) \mid a \in A_{k+1} \}$. As we noticed above non-symmetrical numbers are sliced into groups of two or more, so $|B_k| < \frac{1}{2}|A_k| < \frac{9}{2} \cdot 10^{k-1}$ and $|B_{k+1}| < \frac{1}{2}|A_{k+1}| < \frac{3}{2} \cdot 10^{k-1}$.

Thus, $|G_k| \leq |B_k| + |B_{k+1}| < 6 \cdot 10^{k-1}$, implying $|E_k| = |N_k| - |G_k| > 3 \cdot 10^{k-1}$.

Theorem 1 is proved. We can see that final estimation has a big margin for an error.

5. Stationary numbers. If $a \in N$ and $a = M(a)$ then *a* is called *stationary*. Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 24, 36, 48, 102 happen to be stationary. It is clear that a stationary number is always *m*-digitadditon, since its *m*-generator is itself. Every stationary number a satisfies the equation

$$2a = \hat{a} + s(a) \quad (1)$$

By F_k, $k \geq 1$ we'll denote a set of *k*-digid stationary numbers. We will find all the stationary numbers less than 10^6.

Theorem 2. Let $1 \le k \le 6$, then

$$F_1 = \{\, 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 ,\ F_2 = \{\, 12,\ 24,\ 36,\ 48 \,\},\ F_3 = \{\, 102,\ 204,\ 306,\ 408 \,\}.$$

$$F_4 = \{\, 1002,\ 2004,\ 3006,\ 4008,\ 1372,\ 2374,\ 3376,\ 4378,\ 1743,\ 2745,\ 3747,\ 4749 \,\}$$

$$F_5 = \{\, 10002,\ 20004,\ 30006,\ 40008,\ 17043,\ 27045,\ 37045,\ 47049 \,\}$$

$$F_6 = \{\, 100002,\ 200004,\ 300006,\ 400008,\ 170043,\ 270045,\ 370047,\ 470049 \,\}.$$

By looking at the sets F_5 and F_6 one can deduce an analogy to build some stationary numbers for $k \ge 7$.

Theorem 3. For any $k \ge 4$ the following eight numbers are stationary:

$$c_{i,k} = (\alpha \underbrace{0...0}_{k-2} \beta),\ \alpha = i, \beta = 2i, 1 \le i \le 4,\ e_{j,k} = \left(\gamma 7 \underbrace{0...0}_{k-2} 4\theta \right),\ \gamma = j,\ \theta = 2j+1,\ 1 \le j \le 4.$$

Proof. Theorem 3 is easily proved by plugging the values into equation (1).

Let $H_k = \{c_{i,k}; e_{j,k}\}$. We have $H_k \subset F_k$, but for $k \ge 7$ the set F_k can hold additional numbers, not lying in H_k. Denote $F_k \setminus H_k = V_k$ for $k \ge 7$, then we get $H_k \cup V_k = F_k$.

Let $a = (\alpha_{k-1} \dots \alpha_0),\ \alpha_{k-1} \ne 0$. The equation (1) writes as follows:

$$2\alpha_{k-1} \cdot 10^{k-1} + \dots + 2\alpha_2 \cdot 10^2 + 2\alpha_1 \cdot 10 + 2\alpha_0 = 2\alpha_0 \cdot 10^{k-1} + \dots + \alpha_{k-3} \cdot 10^2 + \alpha_{k-2} \cdot 10 + 2\alpha_{k-1} + s(a)$$

(2) We can see that $1 \le \alpha_{k-1} \le 4$ and $\alpha_0 = 2\alpha_{k-1}$ or $\alpha_0 = 2\alpha_{k-1} + 1$. Let

$$10\alpha_{k-2} + \alpha_{k-1} + s(a) = 20\alpha_1 + 2\alpha_0 + \Delta. \quad (3)$$

Then $\Delta = 1\ \alpha_{k-2} + 2\alpha_{k-1} - 9\ \alpha_1 - \alpha_0 + (\alpha_{k-3} + \dots + \alpha_3)$ (4)

The definition of Δ and equation (2) imply that $\Delta = l \cdot 10^2$, where $l = -1,\ 0,\ 1,\ 2, \dots$. After plugging the expression (3) into equation (2) and dividing (2) by 10^2 we get

$$2\alpha_{k-1} \cdot 10^{k-3} + 2\alpha_{k-2} \cdot 10^{k-4} + 2\alpha_{k-3} \cdot 10^{k-5} + \dots + 2\alpha_2 = \quad (5)$$
$$= \alpha_0 \cdot 10^{k-3} + \alpha_1 \cdot 10^{k-4} + \alpha_2 \cdot 10^{k-5} + \dots + \alpha_{k-3} + l$$

Next we find the variables by pairs: $\{\alpha_{k-3},\ \alpha_2\}$ first, then $\{\alpha_{k-4},\ \alpha_3\}$ and so on.

Theorem 4. Let $k \ge 7$, $\Delta = l \cdot 10^2$, where $-1 \le l \le 9$. For the pair $\{\alpha_{k-3}, \alpha_2\}$ we have 19 following possibilities:

№	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
l	-1	-1	0	0	1	1	2	2	3	3	4	4	5	6	6	7	8	9	9
α_{k-3}	3	9	0	6	3	7	4	0	1	7	4	8	1	2	8	9	2	3	9
α_2	6	9	0	3	7	4	8	1	2	5	9	6	3	4	7	8	5	6	9

Proof. From the equation (5) we have the following systems of equations that contain variables α_{k-3} and α_2:

1) $\begin{cases} 2\alpha_{k-3} = \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l \end{cases}$

2) $\begin{cases} 2\alpha_{k-3} = \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l + 10 \end{cases}$

3) $\begin{cases} 2\alpha_{k-3} + 1 = \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l \end{cases}$

4) $\begin{cases} 2\alpha_{k-1} + 1 = \alpha_2 \\ \alpha_2 = \alpha_{k-3} + l + 10 \end{cases}$

5) $\begin{cases} 2\alpha_{k-3} = 10 + \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l \end{cases}$

6) $\begin{cases} 2\alpha_{k-3} = 10 + \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l + 10 \end{cases}$

7) $\begin{cases} 2\alpha_{k-3} + 1 = 10 + \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l \end{cases}$

8) $\begin{cases} 2\alpha_{k-3} + 1 = 10 + \alpha_2 \\ 2\alpha_2 = \alpha_{k-3} + l + 10 \end{cases}$

By solving those systems we can get all the aforementioned solutions $\{\alpha_{k-3}, \alpha_2\}$. *Theorem 4* is proved.

Thus, we found all the pairs $\{\alpha_{k-3}, \alpha_2\}$. Then we will find the pair $\{\alpha_{k-4}, \alpha_3\}$ and so forth. For every of 19 possibilities for the pair $\{\alpha_{k-3}, \alpha_2\}$ we must solve 4 systems of equations to find $\{\alpha_{k-4}, \alpha_3\}$. Those 19 possibilities all fall into one of the following 4 types, such that all the solutions of the same type lead to the same values of $\{\alpha_{k-4}, \alpha_3\}$:

$T_1 = \{2\alpha_{k-3} \equiv \alpha_2 (\bmod 10)$ and $\alpha_2 \le 4\}$ includes 3, 6, 9, 14 solutions,
$T_2 = \{2\alpha_{k-3} \equiv \alpha_2 (\bmod 10)$ and $\alpha_2 \ge 5\}$ includes 1, 7, 12, 16, 18 solutions,
$T_3 = \{2\alpha_{k-3} + 1 \equiv \alpha_2 (\bmod 10)$ and $\alpha_2 \le 4\}$ includes 4, 8, 13 solutions,
$T_4 = \{2\alpha_{k-3} + 1 \equiv \alpha_2 (\bmod 10)$ and $\alpha_2 \ge 5\}$ includes 2, 5, 10, 11, 15, 17, 19 solutions

In case of T_1 we get $\alpha_{k-4} = 0$, $\alpha_3 = 0$. A pair $\{0, 0\}$ also falls into T_1 giving the same values $\alpha_{k-5} = 0$, $\alpha_4 = 0$ again and so on. Thus, in T_1 we have $\alpha_{k-4} = \alpha_{k-5} = \ldots = \alpha_4 = \alpha_3 = 0$.

In case of T_2 or T_3 we get $\alpha_{k-4} = 6$, $\alpha_3 = 3$. A pair $\{6, 3\}$ falls into T_2 as well, giving $\alpha_{k-5} = 6$, $\alpha_4 = 3$ again and so forth. This sequence leads to contradiction in the middle of a. Thus, cases T_2 and T_3 give us no solutions.

In case T_4 we get $\alpha_{k-4} = 9$, $\alpha_3 = 9$. A pair $\{9, 9\}$ is also in T_4. And thus, in T_4 we have $\alpha_{k-4} = \alpha_{k-5} = \ldots = \alpha_4 = \alpha_3 = 9$.

Shortly speaking, now we must consider possibilities 3, 6, 9, 14 (type T_1) and 2, 5, 10, 11, 15, 17, 19 (type T_4). Let's start from possibility number 2.

Theorem 5. Let $k \geq 7$, $\Delta = -100$ and $a = (a_{k-1}\ a_{k-2}\ \underbrace{9...9}_{k-4}\ a_1 a_0)$. Then the set of stationary numbers is $V_7 = \{ h_{i,7} = (\alpha 49999\beta\ )$, where $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$.

Proof. From the statement we have

$\Delta = -100 = 11\alpha_{k-2} + 2\alpha_{k-1} - 19\alpha_1 - \alpha_0 + 9(k-4$. (6)

Since $\alpha_{k-3} = 9 \geq 5$ we have the following systems of equations to consider.

1) $\begin{cases} 2\alpha_{k-2} + 1 = \alpha_1 \\ 2\alpha_{k-1} = \alpha_0 \end{cases}$ 2) $\begin{cases} 2\alpha_{k-2} + 1 = \alpha_1 + 10 \\ 2\alpha_{k-1} + 1 = \alpha_1 \end{cases}$

By plugging equations of the first system into (6) we get $3\alpha_{k-2} = k + 5$. Considering the fact that $k \geq 7$ and $\alpha_{k-2} \leq 4$ we can find the solution: $\alpha_{k-2} = 4$, $\alpha_1 = 7$, $k = 7$. Next, $\alpha_0 = 2\alpha_{k-1}$, where $1 \leq \alpha_{k-1} \leq 4$ and we get the stationary numbers $h_{i,7}$, $1 \leq i \leq 4$.

In case of the system 2) there are no solutions. *Theorem 5* is proved.

Solutions to 10 other possibilities are similar to the considered one, so we will just provide (without proof) the following three theorems.

Theorem 6. In cases 6, 9, 14 and 10, 15, 17, 19 no solutions can be found.

Theorem 7. Let $\Delta = 0$. Then the set of stationary numbers for $k \geq 7$ is identical to the set H_k.

Theorem 8. Considering cases 5 and 11 we can get the following sets of stationary numbers:

$V_{16} = \{ d_{i,16} = (\alpha 03\ \underbrace{9...9}_{10}\ 70\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$,

$Z_{49} = \{ f_{i,49} = (\alpha 04\ \underbrace{9...9}_{44}\ 70\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$,

$Z_{52} = \{ f_{i,52} = (\alpha 14\ \underbrace{9...9}_{47}\ 2\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$,

$Z_{55} = \{ f_{i,55} = (\alpha 24\ \underbrace{9...9}_{50}\ 4\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$,

$V_{58} = \{ d_{i,58} = (\alpha 34\ \underbrace{9...9}_{53}\ 6\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$,

$V_{61} = \{ f_{i,61} = (\alpha 44\ \underbrace{9...9}_{57}\ 8\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \leq i \leq 4\ \}$,

$V_{43} = \{ f_{i,43} = (\alpha 54\ \underbrace{9...9}_{38}\ 0\beta)$, где $\alpha = i$, $\beta = 2i + 1$, $1 \leq i \leq 4\ \}$,

$V_{46} = \{ f_{i,46} = (\alpha 64\ \underbrace{9...9}_{41}\ 2\beta)$, где $\alpha = i$, $\beta = 2i + 1$, $1 \leq i \leq 4 \}$,

$W_{49} = \{ f_{i,49} = (\alpha 74\ \underbrace{9...9}_{44}\ 4\beta)$, где $\alpha = i$, $\beta = 2i + 1$, $1 \leq i \leq 4\ \}$,

$W_{52} = \{ f_{i,52} = (\alpha 84\ \underbrace{9...9}_{47}\ 6\beta)$, где $\alpha = i$, $\beta = 2i+1$, $1 \le i \le 4$ $\}$,

$W_{55} = \{ f_{i,55} = (\alpha 94\ \underbrace{9...9}_{50}\ 8\beta)$, где $\alpha = i$, $\beta = 2i+1$, $1 \le i \le 4$ $\}$.

Thus, we found all the stationary numbers, when $k \ge 7$ and $\Delta = l \cdot 10^2$, where $-1 \le l \le 9$.

In the case of $k \ge 7$ and $\Delta = 10^3$ stationary can also be found by the same algorithm. We will simply provide the results in theorem 9.

Theorem 9. If $k \ge 7$ and $\Delta = 10^3$, then all the stationary numbers can be found in the following sets:

$V_{118} = \{ q_{i,118} = (\alpha 003\ \underbrace{9...9}_{110}\ 700\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \le i \le 4$ $\}$,

$V_{121} = \{ q_{i,121} = (\alpha 103\ \underbrace{9...9}_{113}\ 702\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \le i \le 4$ $\}$,

$V_{124} = \{ q_{i,124} = (\alpha 203\ \underbrace{9...9}_{116}\ 704\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \le i \le 4$ $\}$,

$V_{127} = \{ q_{i,127} = (\alpha 303\ \underbrace{9...9}_{119}\ 706\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \le i \le 4$ $\}$,

$V_{130} = \{ q_{i,130} = (\alpha 403\ \underbrace{9...9}_{122}\ 708\beta)$, где $\alpha = i$, $\beta = 2i$, $1 \le i \le 4$ $\}$.

For every a with a rank $d(a) \le 130$ we have $\Delta < 1100$. Denote $R = \{ 7, 16, 43, 46, 49, 52, 55, 58, 61, 118, 121, 124, 127, 130 \}$, $Q = \{ 49, 52, 55 \}$.

Summing up all the results of statements $3-9$ we can formulate the following theorem.

Theorem 10. Let $7 \le k \le 130$, then

а) if $k \in R$, then $F_k = H_k \cup V_k$,

б) if $k \in Q$, then $F_k = H_k \cup Z_k \cup W_k$,

в) if $k \notin (R \cup Q)$, then $F_k = H_k$.

Let's show now how to find stationary numbers is a general case. First consider the case $\Delta = l_1 \cdot 10^2 + l_2 \cdot 10^3$, where $l_1, l_2 \in I$. First, we should make systems of equations with $\{ \alpha_{k-4}, \alpha_3 \}$ to find their values. Next, using the values $\{ \alpha_{k-4}, \alpha_3 \}$ make systems of equations to find values $\{ \alpha_{k-3}, \alpha_2 \}$. After that using the values $\{ \alpha_{k-3}, \alpha_2 \}$ make systems of equations with variables $\{ \alpha_{k-2}, \alpha_1 \}$ and $\{ \alpha_{k-1}, \alpha_0 \}$ and then plug these values into (4). If the resulting systems have solutions we will find the stationary numbers.

To give an example, when $\Delta = 1300$, stationary numbers, not lying in the set H_k, will have the values $t_{i,151} = (\alpha 013\ \underbrace{9...9}_{143}\ 720\beta)$, where $\alpha = i$, $\beta = 2i$, $1 \le i \le 4$.

In every particular case $\Delta = l_1 \cdot 10^2 + l_2 \cdot 10^3 + ... + l_m \cdot 10^{m+1}$, where $l_i \in I$, we'll be able to find the values of pairs $\{\alpha_{k-m-2}, \alpha_{m+1}\}$, $\{\alpha_{k-m-1}, \alpha_m\}$, etc. In those cases, where there is a solution, we'll find the values of stationary numbers.

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