Research, Surveys, Results /
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COMPARATIVE ANALYSIS REGARDING THE USE OF COMPLEX NUMBERS IN SECONDARY SCHOOL

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Abstract. The study of Euclidean plane geometry is compulsory for high school students. Practice shows that even the most advanced students face difficulties when learning this material, and especially the content related to the metric characteristics of the geometric figures and their use. On the other hand, the high school syllabus also includes complex numbers, characterized by an analytical apparatus, which is adequate for learning Euclidean plane geometry. This paper presents the results of a comparative analysis of the scores of the students related to the metric characteristics of the geometric figures in the Euclidean plane and their use, when studied with the use of complex numbers (Malcheski, Grozdev & Anevska, 2015), and when studied in a standard way (Mitrović & al., 1998).
Keywords: complex number, triangle, circle, test, hypothesis.

1. Introduction

The modern education systems are faced to the challenge of differentiation and integration of the mathematics instruction. As we know, the division of mathematics to numerous scientific, and naturally instructional disciplines is one of the reasons for the existing differentiation of mathematics instruction. This is why there have been tendencies lately both for intra-subject and inter-subject integration of mathematics instruction. There is also a tendency to achieve the integration of mathematics instruction by changing the syllabi, i.e. by revising them, all with the goal to:

– improve inter-subject integration of mathematics instruction,

– stimulate the students to acquire permanent and applicable, operative and structural knowledge, and

– improve the training of the students for successful inclusion in the higher levels of education.

Accomplishing the mentioned goals is a comprehensive and complex task, which requires analysis of the entire material covered in the high school education, and this cannot be realized by a single research. Having in mind that the students have continuously displayed very poor results when dealing with transformations in the Euclidean plane, as well as the metric characteristics of the plane geometric figures and their use, we developed a syllabus which is based on the use of the complex numbers, and can be listed as an elective subject in comprehensive high school, which is the subject matter of the book (Malcheski, Grozdev & Anevska, 2015). Using this syllabus and the existing syllabi (Mitrović & al., 1998), we carried out an experiment with the purpose to fulfill the goals mentioned previously. The results of the experiment and the conclusions which refer to learning transformations in the Euclidean plane are presented in (Anevska, Grozdev & Malcheski, 2015), while (Anevska, Gogovska & Malcheski, 2015) and (Anevska, 2014) present some views about certain questions related to the application of complex numbers when learning this content. Further on in the paper we present the results from the experiment, which answer the following question:

Compared to the existing syllabi, does acquiring knowledge and skills about the metric characteristics of the geometric figures and their use by applying complex numbers in the high schools stimulate acquiring advanced operative and structural knowledge?

2. Research design

The previously mentioned question defines the subject of the research, which is connected with the scores of the students regarding the metric characteristics of the geometric figures and their use by following the experimental syllabus, compared to the scores when the existing syllabi are followed. We set the following hypothesis according to the subject of our research:

> *The experimental syllabus results in greater knowledge and skills as opposed to the existing syllabi regarding the metric characteristics of the geometric figures and their use.*

As a result of the inability to get a simple random sample, in the period between January 20 and May 20, 2014, we carried out an experiment with voluntary participation of two groups, a control group and an experimental one, each consisting of 25 students with advanced mathematical knowledge and skills from a high school of science. The experiment had the following stages:

– division of the students in a control and an experimental group, the basic criterion being that the students have approximately equal scores in mathematics in the previous school years,

– the control group, using book (Mitrović & al., 1998) and additional literature, revised material related to the metric characteristics of the plane geometric figures, with emphasis on triangles and circles,

– the experimental group, using book (Malcheski, Grozdev & Anevska, 2015),

had instruction which used complex numbers to elaborate the topic Geometry of the circle and the triangle,
– after the instruction, a test was given to assess the knowledge of the students, resulting in an analysis of the results, including:
i) assessment of the validity of the test, i.e. do the scores of the students from the control and experimental group follow normal distribution, and
ii) comparison of the scores of the students from the control and experimental group, carried out by testing the hypotheses referring to the comparison of the mathematical expectations and the distributions of the scores of the students of the control and experimental group.

3. Research results

As we have already mentioned, the test was created with the purpose to assess the knowledge of the students from the control and experimental group regarding the topic Geometry of the circle and the triangle, covered by both syllabi. The test contained 6 tasks and the students were given 90 minutes to complete it. The scores of the students were assessed according to a proportional scale, since it allows the use of the Kolmogorov-Smirnov test that is used to test the hypothesis if the scores of the students have an adequate normal distribution, i.e. it evaluates the quality of the test. Further on, we compared the scores of the students from the control and the experimental group for every task and we gave adequate comments. Below we present the test which was taken by the students from the two groups.

TEST

1. (15 points). If A_1, B_1 and C_1 are the midpoints of the sides BC, AC and AB of ΔABC, prove that

$$A_1B_1 \parallel AB, \quad B_1C_1 \parallel BC, \quad C_1A_1 \parallel CA,$$
$$2\overline{A_1B_1} = \overline{AB}, \quad 2\overline{B_1C_1} = \overline{BC}, \quad 2\overline{C_1A_1} = \overline{CA}.$$

2. (20 points). If H and O are the orthocenter and center of the circumscribed circle around ΔABC, then

$$\overline{OH}^2 = 9R^2 - (\overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2),$$

where R is the length of the radius of the circumscribed circle. Prove it!

3. (20 points). S is the center of the circumscribed circle, and H is the orthocenter of ΔABC. Point Q is such that it makes S the midpoint of the segment HQ. T_1, T_2 and T_3 are the centroids of $\Delta BCQ, \Delta CAQ$ and ΔABQ respectively. Prove that

$$\overline{AT}_1 = \overline{BT}_2 = \overline{CT}_3 = \tfrac{4}{3} R,$$

where R is the radius of the circumscribed circle of ΔABC.

4. (15 points). ΔABC has points D and E on sides BC and CA respectively, in such a way that $\overline{BD} = \overline{CE} = \overline{AB}$. We draw a straight line (l) parallel to AB through point D. If $M = (l) \cap BE$ and $F = CM \cap AB$, then $\overline{AB}^3 = \overline{AE} \cdot \overline{FB} \cdot \overline{CD}$. Prove it!

5. (15 points). There is a circumscribed circle around the equilateral triangle ABC. The random point M belongs to the arc $\overset{\frown}{BC}$ to which the point M does not belong. Prove that $\overline{BM} + \overline{CM} = \overline{AM}$.

6. (15 points). There is a cyclical quadrilateral $ABCD$. Points P and Q are symmetrical to point C when compared to the straight lines AB and AD respectively. Prove that the straight line PQ passes through the orthocenter of the triangle ABD.

Table 1 presents the results of the scores of the control group students. Since we do not have information about the arithmetic mean and the mean square deviation, and since this information is required for further analysis, we will calculate them from the data presented in Table 1. The arithmetic mean, i.e. the average number of points scored by the students is $\overline{x}_{25} = 53,8$, making the mean square deviation $\overline{s}_{25} = 13,51$.

Table 1. Scores of the control group students

Student	Points per task					
	1	2	3	4	5	6
1.	15	5	0	0	10	0
2.	15	5	0	5	5	5
3.	15	5	5	0	5	5
4.	15	5	0	10	10	0
5.	15	15	5	5	0	0
6.	15	5	10	0	5	5
7.	15	10	5	0	5	10
8.	15	5	10	0	15	0
9.	15	10	10	5	5	0
10.	15	10	5	10	0	10
11.	15	5	5	5	10	10
12.	15	5	5	5	10	10

13.	15	5	0	15	10	10
14.	15	5	10	10	15	0
15.	15	10	5	10	5	10
16.	15	10	10	5	10	10
17.	15	5	10	10	15	5
18.	15	5	10	10	15	5
19.	15	10	15	10	15	0
20.	15	10	0	15	10	15
21.	15	10	10	10	10	10
22.	15	10	10	10	10	15
23.	15	15	10	15	15	0
24.	15	20	0	15	10	15
25.	15	15	10	15	15	15

Taking into consideration that the test is valid, objective, reliable and sensitive, i.e. its characteristics are adequate, the scores of the students should follow normal distribution $\mathrm{N}(54;14^2)$. This is an indicator that we should first test H_0 : the function of distribution F_X of the scores of the students is equal to the normal distribution, i.e. the hypothesis $H_0 : F_X = \mathrm{N}(54;14^2)$. For this purpose, as we have already mentioned, we will use the Kolmogorov-Smirnov test with a level of significance $\alpha = 0,05$, where $z_i = \frac{x_i - 54}{14}$. The calculations are presented in Table 2.

According to the data in Table 2, the greatest value of $|F_n(x) - F(x)|$ is $d_{25} = 0,09891$ and it is achieved for $x = 45$. Since the level of significance is $\alpha = 0,05$ and the data number is $n = 25$, from Kolmogorov's criterion table, we established that $d_{25;\ 0,05} = 0,2639$. Since

$$d_{25} = 0,09891 < 0,2639 = d_{25;\ 0,05}$$

we have no reason to dismiss the assumption that the distribution of the scores of the students regarding the first test is $\mathrm{N}(54;14^2)$.

Table 2. Kolmogorov–Smirnov test for the control group

x_i	n_i	$F_n(x_i)$	$z_i = \frac{x_i - 54}{14}$	$F(x_i)$	$\lvert F_n(x) - F(x) \rvert$
30	1	0,04	-1,71	0,04363	0,00363
35	2	0,12	-1,36	0,08691	0,03309
40	3	0,24	-1,00	0,15866	0,08134
45	3	0,36	-0,64	0,26109	0,09891
50	3	0,48	-0,29	0,38591	0,09409

55	3	0,60	0,07	0,52790	0,07210
60	3	0,72	0,43	0,66640	0,05360
65	3	0,84	0,79	0,78524	0,05476
70	2	0,92	1,14	0,87286	0,04714
75	1	0,96	1,50	0,93319	0,02681
85	1	1	2,21	0,98610	0,01390

We will use the Kolmogorov-Smirnov test both for the control and the experimental group. Table 3 presents the scores regarding each task by every student from the experimental group individually.

Table 3. Scores of the experimental group students

Student	Points per task					
	1	2	3	4	5	6
1.	10	5	5	0	10	10
2.	10	5	10	10	10	0
3.	15	10	10	5	5	5
4.	15	5	10	10	10	0
5.	15	15	5	5	0	10
6.	15	10	10	0	10	10
7.	15	10	5	10	10	10
8.	15	10	15	10	5	10
9.	15	15	10	10	10	5
10.	15	10	10	15	15	0
11.	15	5	10	15	10	10
12.	15	10	10	15	10	10
13.	15	10	10	10	15	10
14.	15	20	10	0	15	10
15.	15	5	10	15	10	15
16.	15	10	10	10	10	15
17.	15	20	5	10	10	15
18.	15	10	15	10	15	10
19.	15	15	15	10	15	5
20.	15	20	10	10	15	10
21.	15	15	10	10	15	15
22.	15	15	10	15	15	15
23.	15	20	10	15	10	15
24.	15	20	20	15	10	15
25.	15	20	20	15	15	15

Analogously to the previous cases, the data in Table 3 shows that the arithmetic mean, i.e. the average number of points scored by the students is $\overline{x}_{25} = 68,4$, making the mean square deviation $\overline{s}_{25} = 14,61$. According to this, in order to assess the measuring characteristics of the test, in terms of the experimental group, we need to test the hypothesis H_0 : the function of distribution F_X of the scores of the students is equal to the adequate normal distribution, i.e. the hypothesis $H_0 : F_X = \mathrm{N}(68;15^2)$. For this purpose we will once again use the Kolmogorov-Smirnov test with a level of significance $\alpha = 0,05$, where $z_i = \frac{x_i - 68}{15}$. The calculations are presented in Table 4.

Table 4. Kolmogorov–Smirnov test for the experimental group

x_i	n_i	$F_n(x_i)$	$z_i = \frac{x_i - 68}{15}$	$F(x_i)$	$\lvert F_n(x) - F(x) \rvert$
40	1	0,04	-1,87	0,03074	0,00926
45	1	0,08	-1,53	0,06301	0,01699
50	3	0,2	-1,20	0,11507	0,08493
55	1	0,24	-0,87	0,19215	0,04785
60	1	0,28	-0,53	0,29806	0,01806
65	4	0,44	-0,20	0,42074	0,01926
70	5	0,64	0,13	0,55172	0,08828
75	3	0,76	0,47	0,68082	0,07918
80	2	0,84	0,80	0,78814	0,05186
85	2	0,92	1,13	0,87076	0,04924
95	1	0,96	1,80	0,96407	0,00407
100	1	1	2,13	0,98341	0,01659

According to the data in Table 4, the greatest value of $\lvert F_n(x) - F(x) \rvert$ is $d_{25} = 0,08828$ and it is achieved for $x = 70$. Since the level of significance is $\alpha = 0,05$, and the number of students (data) is $n = 25$, from Kolmogorov's criterion table, we established that $d_{25;\,0,05} = 0,2639$. Since

$$d_{25} = 0,08828 < 0,2639 = d_{25;\,0,05}$$

there is no reason to dismiss the assumption that the distribution of the scores of the students regarding the first test is $\mathrm{N}(68;15^2)$.

Previously, we came to the conclusion that the test scores of the two groups follow normal distribution, which allows us to compare them. As we were able to see, the experimental group students scored 68.4 points on average, and the control group students scored 53.8 points on average. The mean square deviation of the experimental group is approximately 14 points, and the one of the control group is approximately 15 points. This means that the scores of the experimental group students in comparison to the scores of the control group students are higher for 27.14%. This allows us to conclude that the experimental syllabus results in significantly better results. Hence, we can conclude that the use of the experimental syllabus related to the mentioned material, i.e. solving tasks, which emphasize the metric characteristics of the geometric figures, results in improved inter-subject integration of the mathematics instruction in the high schools, as well as better preparation of the students for inclusion in the higher levels of education. This is confirmed even more with the test for the difference of the mathematical expectations of unknown distributions and large samples. This is possible because in the previous analyses we established that the scores of the students in the two groups follow normal distribution. In this case

$$\overline{x}_{25}=68,4,\ \overline{y}_{25}=53,8,\ n_1=n_2=25,\ \overline{s}_x=14,61 \text{ and } \overline{s}_y=13,51.$$

We will test the hypothesis $H_0: m_1 \leq m_2$ as opposed to the alternative hypothesis $H_1: m_1 > m_2$, with a level of significance $\alpha=0,01$

$$\frac{\overline{x}_{n_1}-\overline{y}_{n_2}}{\sqrt{n_2\overline{s}_x^2+n_1\overline{s}_y^2}}\sqrt{n_1 n_2}=\frac{68,4-53,8}{\sqrt{25\cdot 14,61^2+25\cdot 13,51^2}}\sqrt{25\cdot 25}=3,60$$

from the table of normal distribution we can establish that $z_{1-\alpha}=2,33$. The final result is

$$\frac{\overline{x}_{n_1}-\overline{y}_{n_2}}{\sqrt{n_2\overline{s}_x^2+n_1\overline{s}_y^2}}\sqrt{n_1 n_2}=3,60>2,33=z_{1-\alpha}.$$

This means that we should dismiss the hypothesis H_0, i.e. at a level of significance $\alpha=0,01$ we accept that the mathematical expectations related to the scores of the experimental group students are higher than the mathematical expectations related to the scores of the control group students.

Further on, the mean square deviations $\overline{s}_x=14,61$ and $\overline{s}_y=13,51$ differentiate insignificantly. Nevertheless, before making a final decision about whether to accept or dismiss the set hypothesis, we are going to compare the distribution of the scores of the two groups. For this purpose, we are going to use the test for equality of distribution of the two independent normally distributed markings, i.e. we are going to test the hypothesis $H_0:\sigma_1^2=\sigma_2^2$ as opposed to the alternative hypothesis $H_1:\sigma_1^2\neq\sigma_2^2$ with a level of significance $\alpha=0,10$. This indicates that

$n_1 = n_2 = 25$, $\overline{s}_x = 14,61$ and $\overline{s}_y = 13,51$, therefore $\frac{n_1(n_2-1)\overline{s}_x^2}{n_2(n_1-1)\overline{s}_y^2} = 1,169472$. Further on, from the Fisher's distribution table we can see that

$$F_{n_1-1,n_2-1;\frac{\alpha}{2}} = F_{24,24;0,05} = 2,66 \text{ and } F'_{n_2-1,n_1-1;\frac{\alpha}{2}} = F_{24,24;0,05} = 2,66,$$

which means that $F_2 = 2,66$ and $F_1 = \frac{1}{2,66} = 0,38$. Therefore, since

$$F_1 = 0,38 < \frac{n_1(n_2-1)\overline{s}_x^2}{n_2(n_1-1)\overline{s}_y^2} = 1,169472 < 2,66 = F_2,$$

we conclude that there is no reason to dismiss the hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$.

The above mentioned allows us to conclude that we should *accept the set hypothesis*, which means that *regarding the use of acquired knowledge in solving tasks related to the metric characteristics of the geometric figures, the students who followed the experimental syllabus had better scores than the students who followed the existing syllabi.*

Regarding the scores of the students in terms of each task separately, Tables 5 and 6 allow us to conclude that the students who followed the experimental syllabus had better scores than the students who followed the control syllabus. The poor results in task 1 are the result of calculation errors made by two students, which means that the problem is not the syllabus, but the concentration of the students. In terms of the second task, it is evident that the average scores of the experimental group students are almost 45% higher and are the result of the simple analytical apparatus characteristic of the complex numbers for solving this type of tasks. The previously mentioned is especially prominent when it comes to the average scores of the experimental group students in the third task. The average score in this task is almost 66% higher in comparison to the score of the control group students. The similar difference of scored points in the second task can also be seen in the sixth task, where the experimental group students scored a result which is 45% higher than the one scored by the control group students, while the differences are smaller regarding the fourth and fifth task and range around 28% and 17% respectively.

Table 5. Points and average score of points per task

Task	1	2	3	4	5	6
Total number of points	375	215	160	195	235	165
Average by student	15	8,6	6,4	7,8	9,4	6,6

Table 6. Points and average score of points per task						
Task	1	2	3	4	5	6
Total number of points	365	310	265	250	275	245
Average by student	14,6	12,4	10,6	10	11	9,8

The facts presented so far allow us to conclude that when solving metric problems and problems related to the significant points of the triangle, the experimental group students scored significantly better results in almost all tasks. These scores are, above all, a result of the analytical apparatus characteristic of the complex numbers, which allows an effective study of the Euclidean plane geometry when placed correctly in the coordinate system.

4. Conclusion

One of the goals of mathematics instruction is for the students to acquire comprehensive, applicable and permanent mathematical knowledge. Reaching this goal is not a simple task at all, however, we believe that the good differentiation and integration of mathematics instruction is necessary for this.

In the previous analysis we dealt with the results from an experiment about the integration of the content of Complex numbers and Euclidean geometry. In high school education, during this experiment, this integration was realized by introducing the elective subject *Geometry of a complex number*, for which an adequate syllabus was created, also used as the basis for book (Malcheski, Grozdev & Anevska, 2015). The results from the research and the structure of the syllabus allow us to conclude that:

– the experimental syllabus increased the inter-subject integration of the mathematics education, and

– by learning about Geometry of the circle and the triangle with the experimental program, the students acquired advanced operative knowledge and structured skills related to the metric characteristics of the circle and the triangle and their use, and

– the realization of the experimental syllabus improved the readiness of the students to include in the higher degrees of education.

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