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COMPARATIVE ANALYSIS OF THE STUDENTS' SCORES IN SOLVING CONSTRUCTIVE TASKS WHEN GEOMETRY IS STUDIED IN A STANDARD WAY AND WITH THE USE OF COMPLEX NUMBERS

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Abstract. The students learn about complex numbers and Euclidean plane geometry in high school. Nevertheless, this material is separated into units instead of being integrated. (Anevska, 2014) presents methodological aspects about the study of exponential presentation of the complex numbers and (Anevska & al.2015-1) presents the possibility for inter-subject integration of the mathematics instruction in the study of the mentioned topics. Further on, (Anevska & al., 2015-2) and (Anevska & al., 2016) present the results of the comparative analyses of the scores of the students regarding the transformations in the Euclidean plane and the metric characteristics of the geometric figures and their use when studied in the standard way (see (Mitrović & al., 1998)), and when studied with the use of complex numbers (see (Malcheski & al., 2015)). This paper offers analogous analyses regarding the acquired knowledge in these two ways when solving constructive tasks.

Keywords: complex number, geometric construction, test, hypothesis.

Introduction. The study of complex numbers in high school is mostly limited to finding solutions of quadratic equations, elementary knowledge regarding the geometric interpretation of the complex numbers and proving the De Moivre's Theorem. Nevertheless, the possibilities for the application of the complex numbers in the study of Euclidean plane geometry are far greater (see (Malcheski & al., 2015)). Having in mind that

– the term "complex number" on itself is very abstract;

– the apparatus used in the study of geometry with the use of complex numbers is very complex,

we developed a syllabus based on the application of the complex numbers in the study of plane geometry. This syllabus can be offered as an elective subject in high school. (Malcheski & al., 2015)) is compiled according to this syllabus. Using this syllabus and the existing syllabi (see (Mitrović & al., 1998)), we carried out an experiment with the purpose to fulfill the goals mentioned previously. The results of the experiment and the conclusions which refer to learning transformations in the Euclidean plane are presented in (Anevska & al., 2015-2), while (Anevska & al., 2016) presents the same regarding the metric characteristics of the plane geometric figures and their use.

One of the goals of geometry is a continuous increase of the knowledge regarding the geometric notions, which can also be accomplished by solving the so-called constructive tasks. This is why we are going to present the results from the experiment, which answer to the following question:

Compared to the existing syllabi in high school education, does acquiring knowledge and skills regarding plane geometry with the use of complex numbers result in acquiring advanced knowledge needed for solving constructive tasks?

Research design. The previously mentioned question defines the subject of the research, which is connected with the scores of the students when solving constructive tasks with the use of the experimental syllabus, compared to the scores of the students when the existing syllabi are used.

According to the subject of our research we set the following hypothesis:

The existing syllabi result in better scores as opposed to the experimental syllabus regarding the solving of constructive tasks.

As a result of the inability to get a simple random sample, in the period between January 20 and May 20, 2014, we carried out an experiment with voluntary participation of two groups, a control group and an experimental one, each consisting of 25 students with advanced mathematical knowledge and skills from a high school of science. The experiment had the following stages:

– division of the students in a control and an experimental group, the basic criterion being that the students have approximately equal scores in mathematics in the previous school years;

– the control group, using book (Mitrović & al., 1998) and additional literature, revised material related to solving constructive tasks;

– the experimental group, using book (Malcheski & al., 2015), had instruction which used complex numbers to fully acquire the needed knowledge for solving constructive tasks;

– after the instruction, a test was given to assess the knowledge of the students resulting in an analysis of the results, including:

i) assessment of the validity of the test, i.e. do the scores of the students from the control and experimental group follow normal distribution, and

ii) comparison of the scores of the students from the control and the experimental group, carried out by testing the hypotheses referring to the comparison of the mathematical expectations and the distributions of the scores of the students of the control and the experimental group.

Research results. As we have already mentioned, we conducted a test to compare the scores of the students. The test had 5 tasks and the students were given 90 minutes to complete it. The scores of the students were assessed according to a proportional scale, since it allows the use of the Kolmogorov-Smirnov test that is used to test the hypothesis that the scores of the students have an adequate normal distribution, i.e. it evaluates the quality of the test. Further on, we compared the scores of the students from the control and the experimental group for every task and we gave adequate comments. Below we present the test which was passed by the students from the two groups.

TEST

1. (20 points) Given are two circles $K'(o',R')$ and $K''(o'',R'')$ and the line (p). Construct a line (q) parallel to the line (p), in such a way that the circles (K') and (K'') create two equal line segments on it.

2. (20 points) Given are two lines (p) and (q) and the point A. Construct an equilateral triangle ABC, in such a way that $B \in (p)$ and $C \in (q)$.

3. (20 points) Given are two lines (p) and (q) and the point O. Construct a square $ABCD$ with the center in the point O, in such a way that two of vertexes are positioned on the lines (p) and (q) respectively.

4. (20 points) Given are two lines (p) and (q) and the point O. Construct an equilateral triangle ABC with a center in the point O, in such a way that two of its vertexes are positioned on the lines (p) and (q) respectively.

5. (20 points) Construct a circle through the points A and B and tangent to the circle (K_1).

Table 1 presents the scores of the students from the control group.

Table 1. Scores of the students from the control group

Student	Points per task				
	1	2	3	4	5
	20	15	0	10	0
	20	10	0	15	0
	15	15	10	0	10
	15	20	10	0	10
	15	0	15	15	10
	0	20	15	10	10
	20	0	20	0	20

	0	20	20	20	0
	15	0	15	15	15
	20	20	10	15	0
	15	20	15	0	15
	20	20	20	0	5
	0	20	20	15	10
	10	20	20	20	0
	15	15	15	15	10
	20	20	0	20	10
	20	15	15	15	10
	15	20	20	20	0
	0	20	20	20	20
	20	20	20	20	0
	10	20	20	20	10
	15	20	20	20	10
	20	20	20	20	10
	20	20	20	20	10
	20	20	20	20	20

Since we do not have information about the arithmetic mean and the mean square deviation, and since this information is required for further analysis, we will compute them from the data presented in Table 1. The arithmetic mean, i.e. the average number of the points scored by the students is $\overline{x}_{25} = 68,4$, making the mean square deviation $\overline{s}_{25} = 14,05$ .Taking into consideration that the test is valid, objective, reliable and sensitive, i.e. its characteristics are adequate, and the scores of the students follow normal distribution $\mathrm{N}(68;14^2)$. This is an indicator that we should first test H_0 : the function of distribution F_X of the scores of the students is equal to the normal distribution, i.e. the hypothesis $H_0 : F_X = \mathrm{N}(68;14^2)$. For this purpose, as we have already mentioned, we will use the Kolmogorov-Smirnov test with a level of significance $\alpha = 0,05$, where $z_i = \frac{x_i - 68}{14}$. The calculations are presented in Table 2.

Table 2. Kolmogorov-Smirnov test of the control group

x_i	n_i	$F_n(x_i)$	$z_i = \frac{x_i - 68}{14}$	$F(x_i)$	$\lvert F_n(x) - F(x) \rvert$
45	2	0,08	-1,64	0,05050	0,02950
50	1	0,12	-1,29	0,09853	0,02147
55	3	0,24	-0,93	0,17619	0,06381

60	3	0,36	-0,57	0,28434	0,07566
65	4	0,52	-0,21	0,41683	0,10317
70	3	0,64	0,14	0,55567	0,08433
75	2	0,72	0,50	0,69146	0,02854
80	3	0,84	0,86	0,79955	0,04045
85	1	0,88	1,21	0,88686	0,00686
90	2	0,96	1,57	0,94179	0,01821
100	1	1	2,29	0,98899	0,01101

According to the data in Table 2, the greatest value of $|F_n(x) - F(x)|$ is $d_{25} = 0{,}10317$ and it is achieved for $x = 65$. Since the level of significance is $\alpha = 0{,}05$ and the data number is $n = 25$, from Kolmogorov's criterion table, we established that $d_{25;\,0{,}05} = 0{,}2639$. Since

$$d_{25} = 0{,}10317 < 0{,}2639 = d_{25;\,0{,}05}$$

we have no reason to dismiss the assumption that the distribution of the scores of the students regarding the first test is $N(68;14^2)$.

We will also use the Kolmogorov-Smirnov test for the experimental group. Table 3 presents the scores regarding each task by every student from the experimental group individually.

Table 3. Scores of the students from the experimental group

Student	Points per task				
	1	2	3	4	5
	10	15	0	10	0
	20	0	0	15	0
	10	10	10	0	10
	0	20	10	0	10
	15	0	15	0	10
	0	20	15	10	0
	20	0	0	15	15
	10	0	20	20	0
	0	10	10	15	15
	0	20	15	0	15
	20	15	0	15	0
	20	20	0	10	5

	0	20	20	10	10
	10	20	20	0	10
	20	15	0	15	10
	20	20	10	0	10
	20	10	10	10	10
	10	20	20	20	0
	0	10	20	20	20
	20	20	20	10	0
	10	20	20	15	10
	15	15	15	20	10
	20	20	10	20	10
	20	20	20	15	10
	20	20	15	10	20

The data in Table 3 shows that the arithmetic mean, i.e. the average points scored by the students is $\overline{x}_{25} = 58,2$, making the mean square deviation $\overline{s}_{25} = 14,76$. According to this, in order to assess the measuring characteristics of the test, in terms of the experimental group, we need to test the hypothesis H_0: the function of distribution F_X of the scores of the students is equal to the adequate normal distribution, i.e. the hypothesis $H_0 : F_X = \mathrm{N}(58;15^2)$. For this purpose we will once again use the Kolmogorov-Smirnov test with a level of significance $\alpha = 0,05$, where $z_i = \frac{x_i - 58}{15}$. The calculations are presented in Table 4.

Table 4. Kolmogorov-Smirnov test for the experimental group

x_i	n_i	$F_n(x_i)$	$z_i = \frac{x_i - 58}{15}$	$F(x_i)$	$\lvert F_n(x) - F(x) \rvert$
35	2	0,08	-1,53	0,06301	0,01699
40	3	0,20	-1,20	0,11507	0,08493
45	1	0,24	-0,87	0,19215	0,04785
50	4	0,40	-0,53	0,29806	0,10194
55	2	0,48	-0,20	0,42074	0,05926
60	5	0,68	0,13	0,55172	0,12828
70	3	0,80	0,80	0,78814	0,01186
75	2	0,88	1,13	0,87076	0,00924
80	1	0,92	1,47	0,92922	0,00922
85	2	1	1,80	0,96407	0,03593

According to the data in Table 4, the greatest value of $|F_n(x)-F(x)|$ is $d_{25}=0{,}12828$ and it is achieved for $x=60$. Since the level of significance is $\alpha=0{,}05$, and the number of data is $n=25$, from Kolmogorov's criterion table we established that $d_{25;\,0{,}05}=0{,}2639$. Since

$$d_{25}=0{,}12828<0{,}2639=d_{25;\,0{,}05}$$

there is no reason to dismiss the assumption that the distribution of the scores of the students regarding the first test is $N(58;15)$.

Previously, we came to the conclusion that the test scores of the two groups follow normal distribution, which allows us to compare them. As we can see, the control group students scored 68.4 points on average, and the experimental group students scored 58.2 points on average. The mean square deviation of the control group is approximately 14 points, and the one of the experimental group is approximately 15 points. This means that the scores of the control group students in comparison to the scores of the experimental group students are better for 17.53%. This allows us to conclude that the use of the standard syllabus for the study of this material leads to significantly better results. Hence, we can conclude that the use of the control (standard) syllabus results in better scores of the students in tasks including geometric figures in the Euclidean plane and movement and similarities when solving constructive tasks. This is confirmed even more with the test for the difference of the mathematical expectations of unknown distributions and large samples. This is possible because in the previous analyses we established that the scores of the students from the two groups follow normal distribution. In this case

$$\bar{x}_{25}=68{,}4,\quad \bar{y}_{25}=58{,}2,\quad n_1=n_2=25,\quad \bar{s}_x=14{,}05 \text{ and } \bar{s}_y=14{,}76.$$

We will test the hypothesis $H_0: m_1 \le m_2$ as opposed to the alternative hypothesis $H_1: m_1 > m_2$, with a level of significance $\alpha=0{,}01$

$$\frac{\bar{x}_{n_1}-\bar{y}_{n_2}}{\sqrt{n_2\bar{s}_x^2+n_1\bar{s}_y^2}}\sqrt{n_1 n_2}=\frac{68{,}4-58{,}2}{\sqrt{25\cdot 14{,}05^2+25\cdot 14{,}76^2}}\sqrt{25\cdot 25}=2{,}50$$

from the table of normal distribution we can establish that $z_{1-\alpha}=2{,}33$. The final result is

$$\frac{\bar{x}_{n_1}-\bar{y}_{n_2}}{\sqrt{n_2\bar{s}_x^2+n_1\bar{s}_y^2}}\sqrt{n_1 n_2}=2{,}50>2{,}33=z_{1-\alpha}.$$

This means that we should dismiss the hypothesis H_0, i.e. at a level of significance $\alpha=0{,}01$ we accept that the mathematical expectations related to the scores of the control group students are higher than the mathematical expectations related to the scores of the experimental group students.

Further on, the mean square deviations $\overline{s}_x = 14,05$ and $\overline{s}_y = 14,76$ differentiate insignificantly. Nevertheless, before making a final decision about whether to accept or dismiss the set hypothesis, we are going to compare the distribution of the scores of the two groups. For this purpose, we are going to use the test for equality of distribution of the two independent normally distributed markings, i.e. we are going to test the hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ as opposed to the alternative hypothesis $H_0 : \sigma_1^2 \neq \sigma_2^2$ with a level of significance $\alpha = 0,10$. This indicates that $n_1 = n_2 = 25$, $\overline{s}_x = 14,05$ and $\overline{s}_y = 14,76$, therefore

$$\frac{n_1(n_2-1)\overline{s}_x^2}{n_2(n_1-1)\overline{s}_y^2} = 0,906108 .$$

Further on, from the Fisher's distribution table we can see that

$$F_{n_1-1,n_2-1;\frac{\alpha}{2}} = F_{24,24;0,05} = 2,66 \text{ and } F'_{n_2-1,n_1-1;\frac{\alpha}{2}} = F_{24,24;0,05} = 2,66 ,$$

which means that $F_2 = 2,66$ and $F_1 = \frac{1}{2,66} = 0,38$. Therefore, since

$$F_1 = 0,38 < \frac{n_1(n_2-1)\overline{s}_x^2}{n_2(n_1-1)\overline{s}_y^2} = 0,906108 < 2,66 = F_2 ,$$

we can conclude that there is no reason to dismiss the hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$.

The previously mentioned allows us to conclude that we should dismiss the hypothesis, which means that regarding the use of the acquired knowledge in solving constructive tasks, the students who studied according to the existing standard syllabi had better scores than the students who studied according to the experimental syllabus.

Table 5. Points and average score per task

Task	1	2	3	4	5
Total number of points	360	410	380	345	215
Average score by the students	14,4	16,4	15,2	13,8	8,6

Table 6. Points and average score per task

Task	1	2	3	4	5
Total number of points	310	360	295	275	210
Average score by the students	12,4	14,4	11,8	11	8,4

Tables 5 and 6 present the average scores of the students of both the control and the experimental group in terms of each task separately, and we will not discuss them any further. However, we will highlight that in terms of all the five tasks, the scores of the students from the control group were better than the scores of the students from the experimental group. These scores are, above all, a result of the analytical apparatus characteristic of the complex numbers, which allows an effective study of the group properties of the transformations in the Euclidean plane and the metric characteristics of the plane geometric figures (see (Anevska & al., 2015-2), (Anevska & al., 2016) and (Malcheski & al., 2015).

Conclusion. One of the goals of mathematics instruction is for students to acquire comprehensive, applicable and permanent mathematical knowledge. Reaching this goal is not a simple task at all, however, we believe that the good differentiation and integration of mathematics instruction is necessary for this. In the previous analysis we dealt with the results from an experiment about the integration of the content of Complex numbers and Euclidean geometry. In high school education, during this experiment, this integration was realized by introducing the elective subject Geometry of complex number, for which an adequate syllabus was created, also used as the basis for book (Malcheski & al., 2015). The results from the research and the structure of the syllabus allow us to conclude that:

– the experimental syllabus increased the inter-subject and intra-subject integration of the mathematics education;

– the students who study according to the standard syllabus acquire better knowledge and skills for solving constructive tasks from the area of Euclidean geometry, and

– in comparison to the experimental syllabus, the realization of the standard syllabus improved the readiness of the students to be involved in higher degrees of education, where the plane and spatial concepts play an important role (architecture, graphic design, etc.).

The previously mentioned allows us to conclude that it is not necessary to present the elective subject *Geometry of complex number.* Nevertheless, parts of the conducted research, which refer to the transformations in the Euclidean plane and the metric characteristics of the metric figures prove otherwise (see (Anevska & al., 2015-2) and (Anevska & al., 2016)). This is why we need to introduce to students the advantages and disadvantages of each elective subject before they make their final decision.

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