

AN IMPROVEMENT OF THE GERRETSEN'S INEQUALITY FOR NON-OBTUSE TRIANGLES

Šefket Arslanagić
 University of Sarajevo

Abstract. The inequality $s^2 \geq 2R^2 + 8Rr + 3r^2$ is proved in the paper as a generalization of the Gerretsen's inequality $s^2 \geq r(16R - 5r)$ for non-obtuse triangles.

Keywords: non-obtuse triangle, inequality, generalization

In (Bottema et al., 1969) the following inequality 5.14, p. 53 could be found:

$$24Rr - 12r^2 \leq a^2 + b^2 + c^2 \leq 8R^2 + 4r^2, \tag{1}$$

where a, b, c are the lengths of the sides of the triangle $\triangle ABC$ and R and r are the radius of the circumcircle and the radius of the incircle respectively. This is the **Gerretsen's inequality** published in the Dutch journal *Nieuw Tijdschr. Wisk.* 41(1953), 1-7. By the well known equality $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ we rework (1) in the following form:

$$r(16R - 5r) \leq s^2 \leq 4R^2 + 4Rr + 3r^2, \tag{2}$$

where s is the semi-perimeter of the triangle $\triangle ABC$.

In the mathematical literature on inequalities, this inequality is named after Gerretsen and is known to be **Gerretsen's inequality**.

In (Chirciu, 2015) the following series of inequalities 2.41., p. 22, are given:

$$s^2 \geq 16Rr - 5r^2 \geq 15Rr - 3r^2 \geq 14Rr - r^2 \geq \frac{27}{2}Rr \geq 13Rr + r^2 \geq 12Rr + 3r^2 \geq$$

$$\geq 11Rr + 5r^2 \geq 10Rr + 7r^2 \geq 9Rr + 9r^2 \geq 18Rr + 11r^2 \geq 7Rr + 13r^2 \geq 6Rr + 15r^2 \geq$$

$$\geq 5Rr + 17r^2 \geq 4Rr + 19r^2 \geq 3Rr + 21r^2 \geq 2Rr + 23r^2 \geq Rr + 25r^2 \geq 27r^2.$$

We will prove the inequality for non-obtuse triangle:

$$s^2 \geq 2R^2 + 8Rr + 3r^2. \quad (3)$$

This inequality is stronger than the inequality $s^2 \geq r(16R - 5r)$ from (2) because:

$$\begin{aligned} 2R^2 + 8Rr + 3r^2 &\geq r(16R - 5r) \\ \Leftrightarrow 2R^2 - 8Rr + 8r^2 &\geq 0 / : 2 \\ \Leftrightarrow R^2 - 4Rr + 4r^2 &\geq 0 \\ \Leftrightarrow (R - 2r)^2 &\geq 0. \end{aligned}$$

The last is obvious.

It follows, that equalities hold true in (2) and (3) iff $a = b = c$, i. e. in the case of an equilateral triangle.

Proof of inequality (2):

By the Cauchy-Buniakowsky-Schwarz's inequality we have:

$$\begin{aligned} \left((\sqrt{a \cos \alpha})^2 + (\sqrt{b \cos \beta})^2 + (\sqrt{c \cos \gamma})^2 \right) \cdot \left(\left(\frac{\cos \alpha}{\sqrt{a}} \right)^2 + \left(\frac{\cos \beta}{\sqrt{b}} \right)^2 + \left(\frac{\cos \gamma}{\sqrt{c}} \right)^2 \right) &\geq (\cos \alpha + \cos \beta + \cos \gamma)^2, \text{ i.e.} \\ (a \cos \alpha + b \cos \beta + c \cos \gamma) \left(\frac{\cos \alpha}{a} + \frac{\cos \beta}{b} + \frac{\cos \gamma}{c} \right) &\geq (\cos \alpha + \cos \beta + \cos \gamma)^2. \quad (4) \end{aligned}$$

Further we apply the sine law:

$$a \cos \alpha + b \cos \beta + c \cos \gamma = 2R \sin \alpha \cos \alpha + 2R \sin \beta \cos \beta + 2R \sin \gamma \cos \gamma =$$

$$= R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 4R \sin \alpha \sin \beta \sin \gamma =$$

$$= \frac{2R \sin \alpha \cdot 2R \sin \beta}{R} \sin \gamma = \frac{2}{R} \cdot \frac{ab}{2} \sin \gamma = \frac{2F}{R},$$

where F is the area of $\triangle ABC$.

Apply the sine law again:

$$\begin{aligned} \frac{\cos \alpha}{a} + \frac{\cos \beta}{b} + \frac{\cos \gamma}{c} &= \frac{\cos \alpha}{2R \sin \alpha} + \frac{\cos \beta}{2R \sin \beta} + \frac{\cos \gamma}{2R \sin \gamma} = \\ &= \frac{1}{2R}(\operatorname{ctg} \alpha + \operatorname{ctg} \beta + \operatorname{ctg} \gamma) \text{ and } \cos \alpha + \cos \beta + \cos \gamma = 1 + \frac{r}{R}. \end{aligned}$$

From (4) it follows, that:

$$\begin{aligned} \frac{2F}{R} \cdot \frac{1}{2R}(\operatorname{ctg} \alpha + \operatorname{ctg} \beta + \operatorname{ctg} \gamma) &\geq \left(1 + \frac{r}{R} \right)^2 \\ \Leftrightarrow \operatorname{ctg} \alpha + \operatorname{ctg} \beta + \operatorname{ctg} \gamma &\geq \frac{(R+r)^2}{F}. \quad (5) \end{aligned}$$

In (Mitrinović et al., 1989) the equality (69), p. 58 is given:

$$ctg\alpha + ctg\beta + ctg\gamma = \frac{s^2 - r^2 - 4Rr}{2sr}. \quad (6)$$

From (5) and (6) we have:

$$\begin{aligned} \frac{s^2 - r^2 - 4Rr}{2sr} &\geq \frac{(R+r)^2}{F} \\ \Leftrightarrow \frac{s^2 - r^2 - 4Rr}{2F} &\geq \frac{R^2 + 2Rr + r^2}{F} \\ \Leftrightarrow s^2 &\geq 2R^2 + 8Rr + 3r^2, \end{aligned}$$

Which is the inequality (3), q.e.d.

Remark 1. The inequality (3) proposed by A. W. Walker in (Walker & Greening, 1972).

Remark 2. In (Mitrinović & Volenec, 1989), p. 58, proofs are given, that $ctg\alpha$, $ctg\beta$, $ctg\gamma$ are the roots of the equality

$$2srt^3 - (s^2 - r^2 - 4Rr)t^2 + 2srt + (2R+r)^2 - s^2 = 0.$$

It follows from here by the Vieta's theorem that:

$$ctg\alpha + ctg\beta + ctg\gamma = \frac{s^2 - r^2 - 4Rr}{2sr}.$$

Remark 3. The following two equalities are well-known (Grozdev, 2005), (Grozdev, 2007):

$$\begin{aligned} |HI|^2 &= 4R^2 + 4Rr + 3r^2 - s^2 \\ \text{and} \\ |GI|^2 &= \frac{1}{9}(s^2 + 5r^2 - 16Rr), \end{aligned}$$

where \dot{u} and G are the orthocentre, the incentre and the centroid of $\triangle ABC$, respectively. The relations $|HI|^2 \geq 0$ and $|GI|^2 \geq 0$ give evidence of the Gerretsen's inequality (2).

NOTES

1. J. C. Gerretsen, (1907 – 1983), dutch mathematician

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✉ **Prof. Dr. Šefket Arslanagić**
Department of Mathematics
Faculty of Mathematics and Natural Sciences
University of Sarajevo
35, Zmaja od Bosne St.
71 000 Sarajevo, Bosnia and Herzegovina
E-mail: asefket@pmf.unsa.ba