Educational Technologies
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THREE INEQUALITIES FOR THE ALTITUDES OF A TRIANGLE

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Abstract. The following three inequalities are considered in the paper $h_a + h_b + h_c \le 4R + r$, $h_a + h_b + h_c \le \frac{a^2 + b^2 + c^2}{2R}$, $h_{ù} + h \;\; + h \;\; \le \frac{2s^2}{3R}$.
Arrangements of inequalities are considered too.
Keywords: triangle, altitude, inequality, generalization, arrangement

We will prove, that:

$$h_{ù} + h \;\; + h \;\; \le 4R + r, \tag{1}$$

$$h_a + h_b + h_c \le \frac{a^2 + b^2 + c^2}{2R}, \tag{2}$$

$$ù_{ù} + \;\; + \;\; \le \frac{2s^2}{3R}, \tag{3}$$

where a,b,c are the sides of a triangle ΔABC; h_a, h_b, h_c are the corresponding altitudes; s is the semiperimeter and R and r are the circumradius and the inradius of the triangle. For the proof of (1) we will prove the inequality:

$$4R + r \ge s\sqrt{3}. \tag{4}$$

What follows is the proof of (4).

Proof: We will use the well-known formulae (see (Grozdev, 2007)):

$$r_a = \frac{F}{s-a},\ r_b = \frac{F}{s-b},\ r_c = \frac{F}{s-c},$$

where r_a, r_b, r_c are the exradii of ΔABC and F is its area.

Using that $F = rs$ and applying the Heron's formula $F = \sqrt{s(s-a)(s-b)(s-c)}$ we get:

$$r_a r_b = \frac{F^2}{(s-a)(s-b)} = \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)} = s(s-c),$$

Analogously, $r_b r_c = s(s-a)$ and $r_a r_c = s(s-b)$. Consequently:

$$r_a r_b + r_b r_c + r_a r_c = s(s-a) + s(s-b) + s(s-c) = s(s-a+s-b+s-c) = s(3s-2s) = s^2, \text{ i.e.}$$

$$r_a r_b + r_b r_c + r_a r_c = s^2. \qquad (5)$$

Further, apply the well-known formula $abc = 4RF$:

$$r_a + r_b + r_c - r = \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} - \frac{F}{s} =$$

$$= F\left[\frac{s-b+s-a}{(s-a)(s-b)} + \frac{s-s+c}{s(s-c)}\right] = F\left[\frac{2s-(a+b)}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right] =$$

$$= F\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right] = cF \cdot \frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)} =$$

$$= cF \cdot \frac{2s^2-(a+b+c)s+ab}{F^2} = c \cdot \frac{ab}{F} = \frac{abc}{F} = \frac{4RF}{F} = 4R, \text{ i.e.}$$

$$r_a + r_b + r_c = 4R + r. \qquad (6)$$

Now, we go back to the inequality (4). Apply the following evident inequality:

$$(r_a + r_b + r_c)^2 \geq 3(r_a r_b + r_b r_c + r_a r_c)$$

$$\left(\Leftrightarrow \frac{1}{2}\left[(r_a - r_b)^2 + (r_b - r_c)^2 + (r_c - r_a)^2\right] \geq 0\right).$$

It follows from the equalities (5) and (6) that:

$$(4R + r)^2 \geq 3 \cdot s^2,$$

and from here:

$4R + r \geq s\sqrt{3}$, q.e.d.

An equality holds in (4) iff ΔABC is equilateral.

For the proof of the inequality (1) we will use also the well-known Euler's inequality:

$$R \geq 2r \ . \tag{7}$$

(Different proofs of this inequality could be found in (Arslanagić, 2005), (Arslanagić, 2008) and (Arslanagić, 2009).)

Also, we will use the following well-known equality:

$$ab + bc + ca = s^2 + r^2 + 4Rr \ , \tag{8}$$

the proof of which could be found in (Arslanagić, 2005) too.

We have from (8) that:

$$h_a + h_b + h_c = \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} = 2F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) =$$

$$= 2 \cdot \frac{abc}{4R} \cdot \frac{ab + bc + ca}{abc} = \frac{s^2 + r^2 + 4Rr}{2R}, \text{ i.e.}$$

$$h_a + h_b + h_c = \frac{s^2 + r^2 + 4Rr}{2R} \ . \tag{9}$$

Now, using the inequality (4) we get:

$$s\sqrt{3} \leq 4R + r$$

$$\Leftrightarrow 3s^2 \leq 16R^2 + 8Rr + r^2 \ , \tag{10}$$

and applying the inequality (7), we have:

$$24R^2 - 6Rr - 3r^2 - \left(16R^2 + 8Rr + r^2\right) =$$

$$= 8R^2 - 14Rr - 4r^2 = 2\left(4R^2 - 7Rr - 2r^2\right) =$$

$$= 2\left(4R + r\right)\left(R - 2r\right) \geq 0 \text{ , i.e.}$$

$$16R^2 + 8Rr + r^2 \leq 24R^2 - 6Rr - 3r^2 \ . \tag{11}$$

It follows from (10) and (11) that:

$$3s^2 \leq 24R^2 - 6Rr - 3r^2$$

$$\Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2 \ . \tag{12}$$

Finally, from (9) and (12) we get:

$$h_a + h_b + h_c = \frac{s^2 + r^2 + 4Rr}{2R} \leq \frac{8R^2 - 2Rr - r^2 + r^2 + 4Rr}{2R}, \text{ i.e.}$$

$$h_a + h_b + h_c \le 4R + r\,,$$

and this is the inequality (1), q.e.d.

An equality holds in (4) iff $h_a = h_b = h_c \Rightarrow a = b = c$, i.e. when ΔABC is equilateral.

Remark 1: We will prove the following generalization of the inequality (1):

$$h_a^{\alpha} + h_b^{\alpha} + h_c^{\alpha} \le 3^{1-\alpha}\left(4R + r\right)^{\alpha}\,, \qquad (13)$$

where $\alpha \in [0,1]$ is real number.

For the purpose we consider the function $f(x) = x^{\alpha}$, where $x > 0$ and $\alpha \in [0,1]$. Since $f''(x) = \alpha(\alpha - 1)x^{\alpha-2} \le 0$, this function is concave. Applying Jensen's inequality and the inequality (1) we get:

$$h_a^{\alpha} + h_b^{\alpha} + h_c^{\alpha} \le 3\left[\frac{1}{3}\left(h_a + h_b + h_c\right)\right]^{\alpha} \le 3\left[\frac{1}{3}\left(4R + r\right)\right]^{\alpha} = 3^{1-\alpha}\left(4R + r\right)^{\alpha}, \text{ q.e.d.}$$

For $\alpha = 1$ we deduce the inequality (1) as a consequence of inequality (13).

In the sequel we deal with the proof of the inequality (2). We have

$$h_a + h_b + h_c = \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} = 2F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) =$$

$$= 2F \cdot \frac{ab + bc + ca}{abc} = \frac{2abc}{4R} \cdot \frac{ab + bc + ca}{abc} = \frac{ab + bc + ca}{2R},$$

and from the well-known inequality

$$ab + bc + ca \le a^2 + b^2 + c^2 \left(\Leftrightarrow \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] \ge 0\right) \text{ we obtain:}$$

$$h_a + h_b + h_c \le \frac{a^2 + b^2 + c^2}{2R}, \text{ q.e.d.}$$

An equality holds in (2) iff $a = b = c$, i.e. when ΔABC is equilateral.

It remains to prove the inequality (3). We have:

$$h_a + h_b + h_c = \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} = 2F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 2F \cdot \frac{ab + bc + ca}{abc}\,.$$

Thus, it is enough to prove that:

$$2F \cdot \frac{ab + bc + ca}{abc} \le \frac{2s^2}{3R}$$

$$\Leftrightarrow 3FR\cdot\frac{ab+bc+ca}{4FR}\le s^2$$

$$\Leftrightarrow 3\left(ab+bc+ca\right)\le 4s^2$$

$$\Leftrightarrow 3\left(ab+bc+ca\right)\le 4\cdot\frac{\left(a+b+c\right)^2}{4}$$

$$\Leftrightarrow 3\left(ab+bc+ca\right)\le\left(a+b+c\right)^2$$

$$\Leftrightarrow a^2+b^2+c^2\ge ab+bc+ca$$

$$\Leftrightarrow \frac{1}{2}\left[\left(a-b\right)^2+\left(b-c\right)^2+\left(c-a\right)^2\right]\ge 0.$$

Since the last inenquality is obvious, we conclude that the inequality (3) is proved. An equality holds in (3) iff $a=b=c$, i.e. when ΔABC is equilateral.

Now we will prove that the inequality (3) is stronger than the inequality (1). It is enough for the purpose to prove the inequality:

$$s^2\le 4R^2+4Rr+3r^2, \qquad (14)$$

which is known in the mathematical literature as Gerretsen's inequality (see the papare of the present author in the same issue).

We will use the formula for the distance between the incentre and the orthocentre of ΔABC:

$$\left|IH\right|^2=2r^2-4R^2\cos\alpha\cos\beta\cos\gamma, \qquad (15)$$

where α,β,γ are the angles of the triangle. (Two prooofs of this formula could be found in (Arslanagić, 2005).) From the well-known the well-known identity for the angles α,β,γ of ΔABC

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2\left(1+\cos\alpha\cos\beta\cos\gamma\right),$$

we obtain:

$$\cos\alpha\cos\beta\cos\gamma=\frac{1}{2}\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)-1.$$

Using the equality $a^2+b^2+c^2=2\left(s^2+r^2-4Rr\right)$ it follows by the sine law that:

$$\cos\alpha\cos\beta\cos\gamma=\frac{1}{8R^2}\left(a^2+b^2+c^2\right)-1\text{, i.e.}$$

$$\cos\alpha\cos\beta\cos\gamma=\frac{1}{4R^2}\left(s^2-r^2-4Rr\right)-1\text{, i.e.}$$

$$4R^2\cos\alpha\cos\beta\cos\gamma=s^2-r^2-4Rr-4R^2. \qquad (16)$$

From (15) and (16) we obtain:

$$|IH|^2=2r^2-\left(s^2-r^2-4Rr-4R^2\right)$$

and from here (because $|IH|^2\geq 0$) it follows that

$$2r^2-\left(s^2-r^2-4Rr-4R^2\right)\geq 0\text{, i.e.}$$

$$s^2\leq 4R^2+4Rr+3r^2,$$

which is the inequality (14), q.e.d.

An equality holds in (14) iff $\alpha=\beta=\gamma=\frac{\pi}{3}\Rightarrow a=b=c$, i.e. when ΔABC is equilateral.

Further we will prove the inequality:

$$\frac{a^2+b^2+c^2}{2R}\leq 4R+r \qquad (17)$$

$$\Leftrightarrow a^2+b^2+c^2\leq 2R\left(4R+r\right)$$

dnd by $a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right)$ we have:

$$2\left(s^2-r^2-4Rr\right)\leq 2R\left(4R+r\right)$$

$$\Leftrightarrow s^2\leq 4R^2+5Rr+r^2. \qquad (18)$$

Since

$$4R^2+4Rr+3r^2\leq 4R^2+5Rr+r^2, \qquad (19)$$

$$\Leftrightarrow Rr\geq 2r^2$$

$$\Leftrightarrow R\geq r \text{ (Euler's inequality),}$$

We conclude easily that the inequality (2) is stronger than the inequality (1), q.e.d.

Finally we will prove that the inequality (3) is stronger than the inequality (2), i.e. that it is valid the following inequality:

$$\frac{2s^2}{3R} \le \frac{a^2+b^2+c^2}{2R} \qquad (20)$$

$$\Leftrightarrow 4s^2 \le 3\left(a^2+b^2+c^2\right)$$

$$\Leftrightarrow 4\cdot\frac{\left(a+b+c\right)^2}{4} \le 3\left(a^2+b^2+c^2\right)$$

$$\Leftrightarrow \left(a+b+c\right)^2 \le 3\left(a^2+b^2+c^2\right)$$

$$\Leftrightarrow a^2+b^2+c^2 \ge ab+bc+ca$$

$$\Leftrightarrow \frac{1}{2}\left[\left(a-b\right)^2+\left(b-c\right)^2+\left(c-a\right)^2\right] \ge 0,$$

The last inequality is true and this ends the proof of the assertion that the inequality (20) is trues, i.e. that the inequality (3) is stronger than inequality (2). Consequently we have:

$$h_a+h_b+h_c \le \frac{2s^2}{3R} \le \frac{a^2+b^2+c^2}{2R} \le 4R+r\cdot$$

More inequalities concerning the altitudes of ΔABC could be found in (Bottema et al., 1969), namely:

$h_a+h_b+h_c \le s\sqrt{3}$, (6.1, p. 60)

$h_a+h_b+h_c \le \frac{9}{2}R$, (6.10, p. 62)

$h_a+h_b+h_c \le 3\left(R+r\right)$, (6.11, p. 62)

$h_a+h_b+h_c \le 2R+5r$, (6.12, p. 62)

$h_a+h_b+h_c \le \frac{2\left(R+r\right)^2}{R}$. (6.13, p. 63).

Including the already proved inequalities (1), (2) and (3) we obtain the following series of inequalities:

$$h_a + h_b + h_c \le \frac{2s^2}{3R} \le \frac{2\left(R+r\right)^2}{R} \le 2R + 5r \le \frac{a^2 + b^2 + c^2}{2R} \le s\sqrt{3} \le 3\left(R+r\right) \le 4R + r \le \frac{9}{2}R .$$

In the series equalities hold iff and only if ΔABC is equilateral.

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