

SOME NUMERICAL SEQUENCES CONCERNING SQUARE ROOTS (PART TWO)

Rosen Nikolaev, Tanka Milkova, Yordan Petkov
University of Economics – Varna (Bulgaria)

Abstract. The present article is a continuation of the homonymous publication in the journal “Mathematics and Informatics”, issue 5, 2017. The theoretical considerations in it cover all cases when calculating the value of a finite and infinite number of nested square radicals.

Keywords: infinite sequence; nested radical; convergence; mathematical induction

In the first part of this research we examined the convergence of the numerical series $c_n = \sqrt{a + bc_{n-1}}$, $c_1 = \sqrt{a}$, where $a > 0$ and $b > 0$. The aim of the second part is to determine conditions for the limit existence including the case 0.

For a starting base and background we will use again some basic skills and knowledge in the field of:

Infinite numerical series;
Convergence of infinite numerical series;
Limits of infinite numerical series.

On this theoretical basis we will apply some specific and non-standard skills for proving scientifically determined dependencies.

Using theorem 1 from Part one (Nikolaev, Milkova & Petkov, 2017) for example we can easily find the value of the expression $L = \sqrt{3 + 2\sqrt{3 + 2\sqrt{3 + \dots}}}$.

Concerning this, a question arises – can we solve the next problem in a similar way?

Determine the value of the expression $L = \sqrt{3 - 2\sqrt{3 - 2\sqrt{3 - \dots}}}$.

The answer seems to be positive and the solution is based on the statements mentioned in the present paper.

We know from theorem 1 that if $t > b > 0$, then $\sqrt{t(t-b) + b\sqrt{t(t-b) + b\sqrt{\dots}}} = t$.

We will formulate and prove the following theorem:

Theorem 2. If $b < 0$ and $L > \frac{b(1-\sqrt{5})}{2}$, then $\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\dots}}} = L$.

It is obvious that $L > 0$.

We examine the numerical series $c_n = \sqrt{L(L-b)+bc_{n-1}}$, with its first element $c_1 = \sqrt{L(L-b)}$.

This numerical series can be divided into two subseries – the first one consisting of the odd elements $\{c_{2k-1}\}_{k=1}^{\infty}$ and the other one of the even elements, i.e. $\{c_{2k}\}_{k=1}^{\infty}$.

We will formulate and prove two more theorems which will help in proving Theorem 2.

Theorem 3. The subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ converges and when $b < 0$ its limit is equal to $L > \frac{b(1-\sqrt{5})}{2}$.

Theorem 4. The subseries $\{c_{2k}\}_{k=1}^{\infty}$ converges and when $b < 0$ its limit is equal to $L > \frac{b(1-\sqrt{5})}{2}$.

To prove theorem 3 in an easier way we will formulate three lemmas.

Lemma 1. The subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ is limited from below.

Proof: We use the principle of mathematical induction.

1. $k = 1, c_1 = \sqrt{L(L-b)}$.

We will prove that $\sqrt{L(L-b)} > L$.

$$\left| \sqrt{L(L-b)} > L \right|^2;$$

$$L(L-b) > L^2;$$

$$-Lb > 0, \text{ since } L > 0, b < 0 \text{ (by condition)}$$

$$\Rightarrow c_1 > L - \text{correct (1)}$$

$$c_3 = \sqrt{L(L-b)+bc_2}$$

We will prove that $c_3 = \sqrt{L(L-b)+bc_2} > L$.

$$\left| \sqrt{L(L-b)+bc_2} > L \right|^2;$$

$$L(L-b)+bc_2 > L^2;$$

$$bc_2 > Lb \text{ ; } b < 0;$$

$$c_2 < L;$$

$$c_2 = \sqrt{L(L-b)+bc_1}.$$

Let us check if $c_2 < L$:

$$c_2 = \sqrt{L(L-b) + bc_1} < L \Leftrightarrow L(L-b) + bc_1 < L^2 \Leftrightarrow bc_1 < Lb.$$

Since $b < 0$, then $c_1 > L$ - correct, concluded from (1).

2. Let us assume that for each $k = 1, 2 \dots n$ it is true that $c_{2k-1} > L$.

3. We will prove that $k = n + 1$ $c_{2(n+1)-1} = c_{2n+1} > L$

$$\text{If } c_{2n+1} = \sqrt{L(L-b) + bc_{2n}} > L,$$

$$c_{2n+1} = \sqrt{L(L-b) + bc_{2n}} > L \Big|^2 \Leftrightarrow L^2 - Lb + bc_{2n} > L^2 \Leftrightarrow bc_{2n} > Lb \Big| : b < 0 \Leftrightarrow c_{2n} < L;$$

$$c_{2n} = \sqrt{L(L-b) + bc_{2n-1}} < L \Big|^2 \Leftrightarrow L^2 - Lb + bc_{2n-1} < L^2 \Leftrightarrow bc_{2n-1} < Lb \Big| : b < 0 \Leftrightarrow c_{2n-1} > L,$$

which is correct according to the assumption in point 2.

In such a way we proved that the subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ is limited from below, i.e. $c_{2k-1} > L$.

Lemma 2. The subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ is decreasing monotonously.

Proof: Once again we will use the principle of mathematical induction.

1. $k = 1$.

We will check whether $c_1 > c_3$.

$$\sqrt{L(L-b)} > \sqrt{L(L-b) + bc_2} \Leftrightarrow L(L-b) > L(L-b) + bc_2 \Leftrightarrow 0 > bc_2.$$

Since $b < 0$, then c_2 must be positive.

$c_2 = \sqrt{L(L-b) + bc_1}$, and for the value under the radical to be defined we need $L(L-b) + b\sqrt{L(L-b)} > 0$.

$$L(L-b) > -b\sqrt{L(L-b)} \Big|^2 \quad -b > 0, L(L-b) > 0;$$

$$\left[L(L-b) \right]^2 > b^2 L(L-b) \Big| : L(L-b) > 0;$$

$$L(L-b) > b^2;$$

$$L^2 - Lb - b^2 > 0 \Leftrightarrow L \in (-\infty; L_1) \cup (L_2; +\infty),$$

where L_1 and L_2 are the roots of the equation $L^2 - Lb - b^2 = 0$:

$$L_{1,2} = \frac{b \pm \sqrt{b^2 + 4b^2}}{2} = \frac{b(1 \pm \sqrt{5})}{2};$$

$$L_1 = \frac{b(1 + \sqrt{5})}{2} < 0; \quad L_2 = \frac{b(1 - \sqrt{5})}{2} > 0$$

$$\Rightarrow L \in \left(-\infty; \frac{b(1 + \sqrt{5})}{2} \right) \cup \left(\frac{b(1 - \sqrt{5})}{2}; +\infty \right).$$

In theorem 2 we stated that $L > \frac{b(1-\sqrt{5})}{2}$, and in such a way we can make the conclusion that $c_1 > c_3$.

2. We assume that $c_{2k-1} > c_{2k+1}$, for each $k = 1, 2 \dots n$.

3. We will prove that this is satisfied also for $k = 2n+1$: $c_{2n+1} > c_{2n+3}$.

$$c_{2n+3} = \sqrt{L(L-b) + bc_{2n+2}} < c_{2n+1} \Leftrightarrow L(L-b) + bc_{2n+2} < c_{2n+1}^2 \Leftrightarrow$$

$$\Leftrightarrow \cancel{L(L-b)} + b\sqrt{L(L-b) + bc_{2n+1}} < \cancel{L(L-b)} + bc_n \mid : b < 0$$

$$\Leftrightarrow \sqrt{L(L-b) + bc_{2n+1}} > c_n = \sqrt{L(L-b) + bc_{2n-1}}$$

$$\Leftrightarrow bc_{2n+1} > bc_{2n-1} \mid : b < 0$$

$$\Leftrightarrow c_{2n+1} < c_{2n-1},$$

which is correct according to the assumption of the induction.

This way we proved that the subseries $\{c_{2k-1}\}_{k=1}^{\infty}$ is decreasing monotonously.

Lemma 3. The limit of the series $\{c_{2k-1}\}_{k=1}^{\infty}$ is equal to L .

Proof: From Lemma 1 and Lemma 2 we can conclude that $\{c_{2k-1}\}_{k=1}^{\infty}$ has a limit denoting it by l . Then:

$$\lim_{k \rightarrow \infty} c_{2k-1} = l \text{ and } \lim_{k \rightarrow \infty} c_{2k+1} = l.$$

From the recurrence

$c_{2k+1} = \sqrt{L(L-b) + bc_{2k}} = \sqrt{L(L-b) + b\sqrt{L(L-b) + bc_{2k-1}}}$, after a limit transition, we receive an irrational equation for l :

$$l = \sqrt{L(L-b) + b\sqrt{L(L-b) + bl}} \Leftrightarrow l^2 - L(L-b) = b\sqrt{L(L-b) + bl} \Leftrightarrow$$

$$\begin{cases} l^2 - L(L-b) \leq 0 \\ L(L-b) + bl \geq 0 \end{cases} \Leftrightarrow \begin{cases} l \in [-\sqrt{L(L-b)}; \sqrt{L(L-b)}] \\ l \in \left(-\infty; \frac{L(L-b)}{-b}\right] \end{cases}.$$

$$\left[l^2 - L(L-b)\right]^2 = b^2(L(L-b) + bl)$$

$$\frac{b(1-\sqrt{5})}{2}$$

If we consider from Theorem 2 that $L > \frac{b(1-\sqrt{5})}{2} > 0$ together with Lemma 1 then the limit l will be positive. After some transformation, the last system turns to be equivalent to:

$$\left| l \in (0; \sqrt{L(L-b)}) \right|$$

$$l^4 - 2(L^2 - Lb)l^2 - b^3l + (L^2 - Lb)(L^2 - Lb - b^2) = 0$$

Then we obtain immediately that $l_1 = L$ and $l_2 = b - L$ are roots of the above mentioned equation. After factorising its left hand side we get the equivalent form:

$$(l - L)(l - b + L)(l^2 + bl + Lb - L^2 + b^2) = 0,$$

and consequently $l_3 = \frac{-b - \sqrt{4L^2 - 3b^2 - 4Lb}}{2}$ and $l_4 = \frac{-b + \sqrt{4L^2 - 3b^2 - 4Lb}}{2}$.

But from $b < 0$ and $L > \frac{b(1 - \sqrt{5})}{2} > 0$, we have

$$l_1 = L \in (0; \sqrt{L(L-b)}], \quad l_2 = b - L < 0, \quad l_3 = \frac{-b - \sqrt{4L^2 - 3b^2 - 4Lb}}{2} < 0$$

and $l_4 = \frac{-b + \sqrt{4L^2 - 3b^2 - 4Lb}}{2} > \sqrt{L(L-b)}$. Hence $l = L$.

So, from the proved Lemma 1, Lemma 2 and Lemma 3 in advance we conclude that the assertion in Theorem 3 is true.

For proving Theorem 4 we will formulate three more lemmas.

Lemma 4. The subseries $\{c_{2k}\}_{k=1}^{\infty}$ is limited from above.

Proof: We will use again the principle of mathematical induction.

$$1. \quad k = 1, \quad c_2 = \sqrt{L(L-b) + bc_1}.$$

We will prove that $c_2 = \sqrt{L(L-b) + bc_1} < L$.

$$\left| \sqrt{L(L-b) + bc_1} < L \right|^2;$$

$$L^2 - Lb + bc_1 < L^2;$$

$$Lb > bc_1, \text{ since } L > 0, b < 0 \text{ (by condition)}$$

$$\Rightarrow c_1 > L - \text{true to Lemma 1.}$$

$$\text{If } k = 2, \quad c_4 = \sqrt{L(L-b) + bc_3}.$$

We will prove that $c_4 = \sqrt{L(L-b) + bc_3} < L$

$$\left| \sqrt{L(L-b) + bc_3} < L \right|^2$$

$$L^2 - Lb + bc_3 < L^2$$

$$bc_3 < Lb \mid b < 0$$

$$c_3 > L - \text{true to Lemma 1.}$$

2. We assume that for each $k = 1, 2 \dots n$ it is true that $c_{2k} < L$.

3. We will prove that if $k = n + 1$ $c_{2(n+1)} = c_{2n+2} < L$.

And if $c_{2n+2} = \sqrt{L(L-b) + bc_{2n+1}} < L$,

$$c_{2n+2} = \sqrt{L(L-b) + bc_{2n+1}} < L \Leftrightarrow L^2 - Lb + bc_{2n+1} < L^2 \Leftrightarrow bc_{2n+1} < Lb \mid : b < 0 \Leftrightarrow c_{2n+1} > L,$$

which can be concluded by Lemma 1.

So we proved that the subseries $\{c_{2k}\}_{k=1}^{\infty}$ is limited from above by the constant L , i.e. $c_{2k} < L$.

Lemma 5. The subseries $\{c_{2k}\}_{k=1}^{\infty}$ is increasing monotonously.

Proof: We will use once again the principle of mathematical induction.

1. $k = 1$.

We will check whether $c_4 > c_2$.

$$\sqrt{L(L-b) + bc_3} > \sqrt{L(L-b) + bc_1} \Leftrightarrow L(L-b) + bc_3 > L(L-b) + bc_1 \Leftrightarrow bc_3 > bc_1.$$

Because of $b < 0$, we have $c_3 < c_1$, which can be concluded by Lemma 2.

2. We assume that $c_{2k} > c_{2k-2}$, for each $k = 1, 2 \dots n$.

3. We will prove that this is also correct for $k = n + 1$: $c_{2n+2} > c_{2n}$.

$$c_{2n+2} = \sqrt{L(L-b) + bc_{2n+1}} > c_{2n} \Leftrightarrow L(L-b) + bc_{2n+1} > c_{2n}^2 = L(L-b) + bc_{2n-1}$$

$$\Leftrightarrow bc_{2n+1} > bc_{2n-1} \mid : b < 0$$

$$\Leftrightarrow c_{2n+1} < c_{2n-1},$$

which is a consequence of Lemma 2.

Hence, Lemma 5 is proved.

Lemma 6. The limit of the series $\{c_{2k}\}_{k=1}^{\infty}$ is equal to L .

Proof: From Lemma 4 and Lemma 5 we can make the conclusion that the subseries $\{c_{2k}\}_{k=1}^{\infty}$ has a limit and denote it by l . Then:

$$\lim_{k \rightarrow \infty} c_{2k} = l.$$

From the recurrence

$c_{2k} = \sqrt{L(L-b) + bc_{2k-1}}$, after a limit transition and applying Theorem 3, we can conclude that

$$l = \lim_{k \rightarrow \infty} c_{2k} = \lim_{k \rightarrow \infty} \sqrt{L(L-b) + bc_{2k-1}} = \lim_{k \rightarrow \infty} \sqrt{L(L-b) + bL} = L.$$

Thus, $l = L$.

Theorem 4 follows by Lemma 4, Lemma 5 and Lemma 6.

Combining Theorem 3 and Theorem 4 we receive the statement of Theorem 2.

Following the definition of the square root, the following conclusions could be made from Theorem 2:

Conclusion 1: If $L(L-b) < 0$, then the expression $\sqrt{L(L-b) + b\sqrt{L(L-b) + b\sqrt{\dots}}}$ is not defined in the set of the real numbers.

Proof: $c_1 = \sqrt{L(L-b)}$ does not exist because $L(L-b) < 0$.

The following conclusion is also obvious.

Conclusion 2: If $b < 0$ and $L(L-b) < 0$, then the expression

$$\sqrt{L(L-b) + b\sqrt{L(L-b) + b\sqrt{\dots}}}$$

is not defined in the set of the real numbers.

Conclusion 3: If $b < 0$ and $L < \frac{b(1-\sqrt{5})}{2}$, then the expression

$$\sqrt{L(L-b) + b\sqrt{L(L-b) + b\sqrt{\dots}}}$$

is not defined in the set of the real numbers.

Proof: The elements of the subseries $\{c_{2k}\}_{k=1}^{\infty}$ have negative values under the square root.

Theorem 5. Let $b < 0$ and $L = \frac{b(1-\sqrt{5})}{2}$. Then

$$\sqrt{L(L-b) + b\sqrt{L(L-b) + b\sqrt{\dots}}} = \begin{cases} -b, & \text{if the number of roots is a finite odd number,} \\ 0, & \text{if the number of roots is a finite even number,} \\ \text{indefinite,} & \text{if the number of roots is infinite.} \end{cases}$$

Proof: We will use again the principle of mathematical induction.

1. $n = 1$.

$$c_1 = \sqrt{L(L-b)} = \sqrt{\frac{b(1-\sqrt{5})}{2} \left(\frac{b(1-\sqrt{5})}{2} - b \right)} = \sqrt{\frac{b(1-\sqrt{5})}{2} \left(\frac{b(-1-\sqrt{5})}{2} \right)} = \sqrt{b^2} = |b|;$$

$$b < 0 \Rightarrow c_1 = -b.$$

$$c_2 = \sqrt{L(L-b) + bc_1} = \sqrt{L(L-b) + b(-b)} = \sqrt{L(L-b) - b^2} = \sqrt{b^2 - b^2} = 0.$$

2. Let us assume that $c_{2n-1} = -b$ and $c_{2n} = 0$ for each $n = 1, 2, \dots, k$.

3. We will prove that for $n = k+1$ it is true that $c_{2k+1} = -b$ and $c_{2k+2} = 0$.

$$c_{2k+1} = \sqrt{L(L-b) + bc_{2k}} = \sqrt{b^2 + b \cdot 0} = \sqrt{b^2} = |b| = -b.$$

$$c_{2k+2} = \sqrt{L(L-b) + bc_{2k+1}} = \sqrt{b^2 - b^2} = 0.$$

If the number of the roots is infinite, then the value of the expression $\sqrt{L(L-b)+b\sqrt{L(L-b)+b\sqrt{\dots}}}$ cannot be determined.

The proved statements complete the possibilities to calculate expressions of the type $\sqrt{a+b\sqrt{a+b\sqrt{\dots}}}$.

Applying the proved statements, we will examine three problems.

Problem 1. Determine the value of the expression $L = \sqrt{3-2\sqrt{3-2\sqrt{3-\dots}}}$.

Solution:

$$b = -2, a = L(L-b) = L(L+2) = 3;$$

$$L^2 + 2L - 3 = 0;$$

$$L_1 = -3, L_2 = 1;$$

$$L = \frac{b(1-\sqrt{5})}{2} = \frac{-2(1-\sqrt{5})}{2} = \sqrt{5} - 1.$$

We compare $L_1 = -3$ and $L_2 = 1$ with $\sqrt{5} - 1$.

Since $b < 0$, $L < \frac{b(1-\sqrt{5})}{2}$ and the number of roots is infinite, by Conclusion

3 we get that the value of the expression does not exist.

Problem 2. Determine the value of the expression $\sqrt{4-2\sqrt{4-2\sqrt{4-2\sqrt{\dots}}}}$, if:

a) the number of the roots is 2017;

b) the number of the roots is 2018.

Solution:

$$b = -2, a = 4 = L(L-b);$$

$$L^2 + 2L - 4 = 0;$$

$$L_1 = -1 - \sqrt{5}, L_2 = -1 + \sqrt{5}.$$

By Theorem 5 we deduce that $c_{2k-1} = -b = 2$ and $c_{2k} = 0$.

a) $c_{2017} = c_{2k-1} = 2$;

b) $c_{2018} = c_{2k} = 0$.

Problem 3. Determine the value of the expression

$$\sqrt{2017-17\sqrt{2017-17\sqrt{2017-17\sqrt{\dots}}}}.$$

Solution:

$$b = -17, L(L - b) = L(L + 17) = 2017;$$

$$L^2 + 17L - 2017 = 0;$$

$$L_{1,2} = \frac{-17 \pm \sqrt{8357}}{2}.$$

$$\text{From } L > 0 \Rightarrow L = \frac{\sqrt{8357} - 17}{2}.$$

$$\text{We compare } \frac{\sqrt{8357} - 17}{2} \text{ with } \frac{-17(1 - \sqrt{5})}{2} = \frac{17\sqrt{5} - 17}{2} = \frac{\sqrt{1445} - 17}{2}$$

$$\Rightarrow \frac{\sqrt{8357} - 17}{2} > \frac{\sqrt{1445} - 17}{2}.$$

$$\text{From Theorem 2 } \Rightarrow L = \frac{\sqrt{8357} - 17}{2}, \text{ i.e.}$$

$$\sqrt{2017 - 17\sqrt{2017 - 17\sqrt{2017 - 17\sqrt{\dots}}}} = \frac{\sqrt{8357} - 17}{2}.$$

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ЧИСЛОВИ РЕДИЦИ, СВЪРЗАНИ С КВАДРАТНИ КОРЕНИ (ВТОРА ЧАСТ)

Резюме. Статията е продължение на едноименната публикация в сп. „Математика и информатика“, бр. 5, 2017 г. Теоретичните разглеждания в нея обхващат всички случаи при пресмятане стойностите на краен и безкраен брой вложени квадратни радикали.

✉ ¹⁾ **Dr. Rosen Nikolaev, Assoc. Prof.**

²⁾ **Dr. Tanka Milkova, Assoc. Prof.**

³⁾ **Dr. Jordan Petkov, Assist. Prof.**

University of Economics – Varna

77, Kniaz Boris I Blvd.

9002 Varna, Bulgaria

E-mail: ¹⁾ nikolaev_rosen@ue-varna.bg,

²⁾ tankamilkova@ue-varna.bg,

³⁾ jr_petkov@ue-varna.bg