Educational Matters
Научно-методически статии

COMPUTER DISCOVERED MATHEMATICS: CONSTRUCTIONS OF MALFATTI SQUARES

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Abstract. By using the computer program "Discoverer" we find 85 different geometric constructions of Malfatti squares. Of them 79 are new constructions, and the other 6 are old constructions.
Keywords: Malfatti squares; geometric construction; triangle geometry; computer discovered mathematics; Euclidean geometry; "Discoverer"

1. Introduction

The Malfatti squares of a triangle are the three squares each with two adjacent vertices on two sides of the triangle and the two remaining adjacent vertices from those of a triangle in its interior. The triangle in the interior is the Malfatti squares triangle. We denote it by $MaMbMc$. See Figure 1.

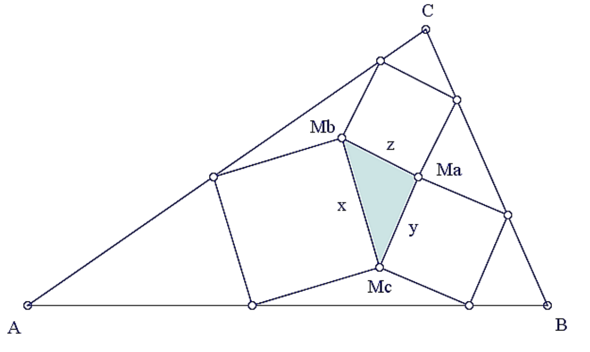

Figure 1

In 1985, Hidetosi Fukagawa (Fukagawa, 1985), Problem 1013. defined the Malfatti squares, named by analogy with Malfatti circles. Fukagawa presented three problems, the first two of which are as follows:

(a) Given a triangle *ABC*, show how to construct its three Malfatti squares.

(b) Given the sides a, b, c of a triangle, calculate the sides x, y, z of its Malfatti squares.

In 1986, Dan Sokolowsky (Sokolowsky, 1986) solved the three problems.

In 2007, Deko Dekov (Dekov, 2007a) published a number of properties of the Malfatti squares triangle, and in (Dekov, 2007b) he gave six new simple constructions of the Malfatti squares. In fact, in (Dekov, 2007b), Dekov presented a new method for geometric constructions. See e.g. (Dekov, 2010), (Grozdev & Dekov, 2016). In 2008, Floor van Lamoen and Paul Yiu (Lamoen & Yiu, 2008) presented a new constriction of the Malfatti squares, and also an alternative construction which repeated one of the construction given in (Dekov, 2007b). Also, Floor van Lamoen and Paul Yiu (Lamoen & Yiu, 2008) gave the barycentric coordinates of the Malfatti squares triangle.

In this paper we prove the six constructions of the Malfatti squares given in (Dekov, 2007b). Then we study again the method described in (Dekov, 2007b) and we find 79 new constructions of the Malfatti squares. By this way we give 79 new relatively simple solutions to Problem 1013 (a) of Fukagawa. Also, we calculate the side lengths of the Malfatti squares triangle and we give a new simple solution to Problem 1013 (b) of Fukagawa.

We use the computer program "Discoverer", created by the authors. In fact, the new method for geometric constructions is applicable, if we use a computer program.

We use barycentric coordinates. See (Grozdev & Dekov, 2016), (Yiu, 2013), (Leversha, 2013). We denote the side lengths of a triangle *ABC* as follows: $BC = a, CA = b$ and $AB = c$. The Kimberling points, given in (Kimberling, ETC), are denoted by X(*n*). The point X(3068) is known as the Malfatti-Moses point. This point is the Centroid of the Malfatti squares triangle. We denote by Δ the area of triangle *ABC*, and we use the Conway's notations (Weisstein, MathWorld):

$$S = 2\Delta, S_A = \frac{b^2+c^2-a^2}{2}, S_B = \frac{c^2+a^2-b^2}{2}, S_C = \frac{a^2+b^2-c^2}{2}$$

The Hatzipolakis triangle of a point $P = (u, v, w)$ is defined by A. P. Hatzipolakis (Hatzipolakis, 2001). The barycentric coordinates of the Hatzipolakis triangle are published by J.-P.Ehrmann (Ehrmann, 2002) as follows:

$$Pa = (uS - \sigma a^2, vS + \sigma S_C, wS + \sigma S_B),$$
$$Pb = (uS + \sigma S_C, vS - \sigma b^2, wS + \sigma S_A),$$
$$Pc = (uS + \sigma S_B, vS + \sigma S_A, wS - \sigma c^2),$$

where $\sigma = u + v + w.$

2. Area and Side Lengths

The following result is published by Floor van Lamoen and Pail Yiu (Lamoen & Yiu, 2008), page 54:

Theorem. The barycentric coordinates of the Malfatti squares triangle $M = MaMbMc$ are as follows:

$$Ma = (S, S + S_A + 2S_C, S + S_A + 2S_B),$$
$$Mb = (S + S_B + 2S_C, S, S + S_B + 2S_B),$$
$$Mc = (S + S_C + 2S_B, S + S_C + 2S_A, S)$$

By using this result, we can easily find the area and the side lengths of the Malfatti squares triangle as follows:

Theorem 1. The area of the Malfatti squares triangle is

$$area = \frac{48\Delta^3}{(6\Delta + a^2 + b^2 + c^2)^2}.$$

Proof. We use the area formula (2) in (Grozdev & Dekov, 2016).

Theorem 2. The side lengths of the Malfatti squares triangle are as follows:

$$x = |MbMc| = \frac{2\Delta\sqrt{2b^2 + 2c^2 - a^2}}{6\Delta + a^2 + b^2 + c^2}$$

$$y = |McMa| = \frac{2\Delta\sqrt{2c^2 + 2a^2 - b^2}}{6\Delta + a^2 + b^2 + c^2}$$

$$z = |MaMb| = \frac{2\Delta\sqrt{2a^2 + 2b^2 - c^2}}{6\Delta + a^2 + b^2 + c^2}.$$

Proof. We use the distance formula (9) in (Grozdev & Dekov, 2016).

Note that Theorem 2 gives a short and simple solution to Fukagawa Problem 1013 (b), (Fukagawa, 1985).

3. Perspective and Homothetic Triangles

In this section we study a few triangles which are homothetic or perspective (but not homothetic) with the Malfatti squares triangle. We use these results in Section 4.

Two triangles are homothetic if their corresponding sides are parallel.

We use the following results: Two lines are parallel if they have the same infinite point ((Yiu, 2013), Section 4.2.2). The infinite point of a line has barycentric coordinates given by the difference of the normalized barycentric coordinates of two distinct points on the line ((Yiu, 2013), Section 4.2.1).

We following Lemma give the infinite points of the side lines of the Malfatti squares triangle.

Lemma 1. The Infinite points $I(MaMb), I(MbMc)$ and $I(McMa)$ of the side lines *MaMb*, *MbMc* and *McMa* of Malfatti squares triangle are as follows:

$$I(MaMb) = (-3a^2 - b^2 + c^2, a^2 + 3b^2 - c^2, 2(a^2 - b^2)),$$
$$I(MbMc) = (2(b^2 - c^2), a^2 - 3b^2 - c^2, -a^2 + b^2 + 3c^2),$$
$$I(McMa) = (-3a^2 + b^2 - c^2, 2(a^2 - c^2), a^2 - b^2 + 3c^2).$$

Based on the above Lemma, we prove that a number of triangles are homothetic with the Malfatti squares triangle.

Theorem 3. The Malfatti squares triangle and the Pedal triangle of the Symmedian point are homothetic. The center of homothety is the Centroid. The ratio of homothety is

$$k = \frac{6\Delta + a^2 + b^2 + c^2}{a^2 + b^2 + c^2}.$$

Proof. The barycentric coordinates of the Pedal triangle of the Symmedian point are as follows:

$$Pa = (0, -c^2 + a^2 + 3b^2, -b^2 + 3c^2 + a^2),$$
$$Pb = (-c^2 + 3a^2 + b^2, 0, -a^2 + b^2 + 3c^2),$$
$$Pc = (-b^2 + c^2 + 3a^2, -a^2 + 3b^2 + c^2, 0).$$

We calculate the infinite points of the side lines of triangle $PaPbPc$ and we see that they coincide with the corresponding infinite points of the side lines of the Malfatti squares triangle. Hence, these triangles are homothetic.

Since the triangles are homothetic, they are perspective. We find the center of homothety as the intersection of lines $MaMb$ and $PaPb$. We use the formula (5) in (Grozdev & Dekov, 2016). We see that the point of intersection, that is, the center of homothety, is the Centroid.

In order to find the ratio of homothety, we calculate the distances $|XPa|$ and $|XMa|$ where X is the center of homothety. We use the distance formula (9) in (Grozdev & Dekov, 2016). Then we find the ratio k as follows:

$$k = \frac{|XPa|}{|XMa|} = \frac{6\Delta + a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$

The Figure 2 illustrates Theorem 3. In figure 2,

– *MaMbMc* is the Malfllati squares triangle,

– *PaPbPc* is the Pedal triangle of the Symmedian point, and

– *X* is the Centroid.

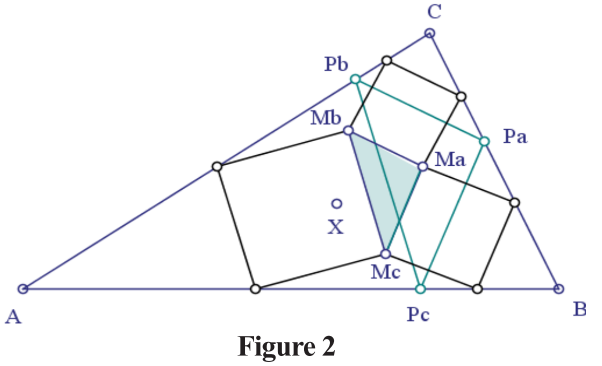

Figure 2

Theorem 4. The Malfatti squares triangle and the Pedal triangle of the Malfatti-Moses point are perspective. The Perspector is the Malfatti-Moses point.

Proof. The barycentric coordinates of the Pedal triangle of the Malfatti-Moses point are as follows:

$$Pa = (0, S_C a^2 + S_C S + a^2 b^2 + a^2 S, S_B a^2 + S_B S + a^2 c^2 + a^2 S)$$
$$Pb = (S_C b^2 + S_C S + a^2 b^2 + b^2 S, 0, S_A b^2 + S_A S + b^2 c^2 + b^2 S),$$
$$Pc = (S_B c^2 + S_B S + a^2 c^2 + c^2 S, S_A c^2 + S_A S + b^2 c^2 + c^2 S, 0),$$

We use formula (3) in (Grozdev & Dekov, 2016) and we find the equations of lines $PaMa, PbMb$ and $PcMc$. Then we use formula (6) in (Grozdev & Dekov, 2016) and we prove that these three lines concur in a point. Finally, we use formula (5) in (Grozdev & Dekov, 2016) and we find the barycentric coordinates of the intersection of lines $PaMa$ and $PbMb$, that is, the perspector. The perspector is the Malfatti-Moses point.

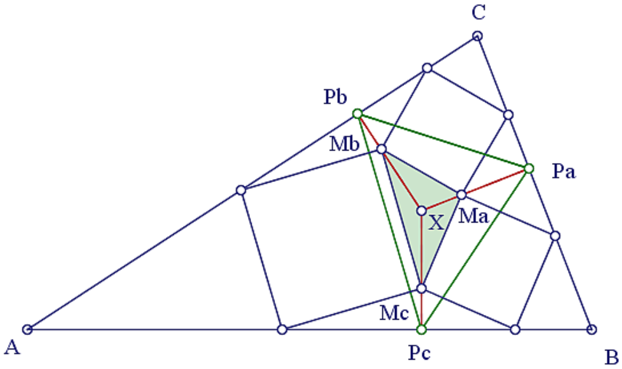

Figure 3

Figure 3 illustrates Theorem 4. In figure 3,

– *MaMbMc* is the Malfatti squares triangle,

– *PaPbPc* is the Pedal triangle of the Malfatti-Moses point, and

– *X* is the Malfatti-Moses point. We see that point *X* is the Perspector of triangles *MaMbMc* and *PaPbPc*.

Theorem 5. Given a point *P*. The Malfatti squares triangle and the Hatzipolakis triangle of point *P* are homothetic.

Proof. The Infinite points of lines PaPb, PbPc and PcPa are the same as the infinite point of the corresponding side lines of triangle MaMbMc.

The next two theorems are special cases of Theorem 5.

Theorem 6. The Malfatti squares triangle and the Hatzipolakis triangle of the Orthocenter are homothetic. The center of homothety is the Inner Kenmotu point. The ratio *k* of homothety is

$$k = \frac{6\Delta + a^2 + b^2 + c^2}{2\Delta}.$$

Proof. The barycentric coordinates of the Orthocenter *H* are $H = (u, v, w) = \left(\frac{1}{S_A}, \frac{1}{S_B}, \frac{1}{S_C}\right)$ We calculate the barycentric coordinates of the

Hatzipolakis triangle of the Orthocenter by using the formulas given in Section 1. The rest of the proof is similar to the proof of Theorem 3.

Theorem 7. The Malfatti squares triangle and the Hatzipolakis triangle of the Centroid are homothetic. The Center of homothety is the Centroid of the Pedal triangle of the Inner Kenmotu point.

Note that the center of homothety in Theorem 7 is not available in (Kimberling, ETC), so that it could be considered as a new notable center.

Problem 1. Find the barycentric coordinates of the center of homothety and the ratio of homothety in Theorem 7.

The Malfatti squares triangle and the Hatzipolakis triangle of arbitrary point *P* are homothetic. Below we give a few additional theorems in whose centers of homotheties are Kimberling points. We encourage the reader to find the ratios of homotheties.

Theorem 8. The Malfatti squares triangle and the Hatzipolakis triangle of the Incenter are homothetic. The center of homotheties is point X(13883).

Theorem 9. The Malfatti squares triangle and the Hatzipolakis triangle of the Circumcenter are homothetic. The center of homothety is point X(7583).

Theorem 10. The Malfatti squares triangle and the Hatzipolakis triangle of the Nagel point are homothetic. The center of homothety is point X(7969).

Theorem 11. The Malfatti squares triangle and the Hatzipolakis triangle of the de Longchamps Point are homothetic. The center of homothety is point X(3070).

Theorem 12. The Malfatti squares triangle and the Hatzipolakis triangle of the Retrocenter are homothetic. The center of homothety is the Symmedian point.

Theorem 13. The Malfatti squares triangle and the Hatzipolakis triangle of the Malfatti-Moses point are homothetic. The center of homothety is the Malfatti-Moses point X(3068).

Theorem 14. The Malfatti squares Triangle and the Triangle of Reflections of the Symmedian point in the Sidelines of Triangle ABC are homothetic. The center of homothety is point X(590).

Theorem 15. The Malfatti squares triangle and the Triangle of Reflections of the Malfatti-Moses Point in the Sidelines of Triangle ABC are homothetic. The center of homothety is point X(3068).

Theorem 16. The Malfatti squares triangle and the Triangle of Reflections of the Malfatti-Moses Point in the Sidelines of the Medial triangle are perspective. The perspector is the Malfatti-Moses point X(3068).

Theorem 17. The Malfatti squares triangle and the Triangle of Reflections of Vertices of the Medial triangle in the Malfatti-Moses point are perspective. The perspector is point X(13639).

4. Geometric Constructions We use the following method (Dekov, 2007b). Define 5-tupe (*T,T1,P1,T2,P2*), where *T; T1* and *T2* are triangles, *P1* is perspector of triangles *T* and *T1*, *P2* is perspector of triangles *T* and *T2* We suppose that *P1 = P2*. If we can construct triangles *T1* and *T2*, and perspectors *P1* and P2, then we can construct triangle *T*.

Suppose that we want to construct the Malfatti squares triangle. By using theorems 3-4 and 6-17, we can find 85 5-tuples able to serve in a construction of the Malfatti squares triangle. Of them 6 constructions are given in (Dekov, 2007b), and 79 are new constructions.

We give two examples.

Construction 1. We use theorems 3 and 14. Denote

– *T* = Malfatti squares triangle,

– *T1* = Pedal triangle of the Symmedian point,

– *P1* = Centroid = perspector of triangles *T* and *T1*,

– *T2* = Triangle of Reflections of the Symmedian point in the Sidelines of Triangle ABC,

– *P2* = point X(590) = perspector of triangles *T* and *T1*.

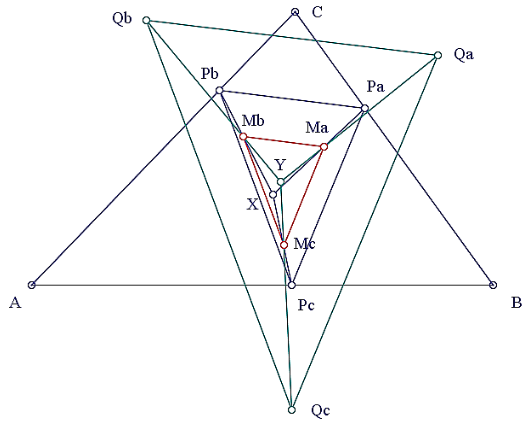

Figure 4

Figure 4 illustrates the construction. In figure 4,

– *PaPbPc* is the Pedal triangle of the Symmedian point,

– *QaQbQc* is the Triangle of Reflections of the Symmedian point in the Sidelines of Triangle ABC,

– *X* is the Centroid, and

– *Y* is the point X(590).

Then

– lines *XPa* and *YQa* concur at point *Ma*,

– lines *XPb* and *YQb* concur at point *Mb*,

– lines *XPc* and *YQc* concur at point *Mc*, and

– triangle *MaMbMc* is the Malfatti squares triangle.

We could construct point X(590) as the intersection of the line through Centroid and Symmedian point and the line through the Nine-Point Center and Inner Kenmotu point.

Construction 2. We use theorems 12 and 13. Denote
– T = Malfatti squares triangle,
– $T1$ = Hatzipolakis triangle of the Retrocenter,
– $P1$ = Symmedian point = perspector of triangles T and $T1$,
– $T2$ = Hatzipolakis triangle of the Malfatti-Moses point,
– $P2$ = Malfatti-Moses point = perspector of triangles T and $T2$.

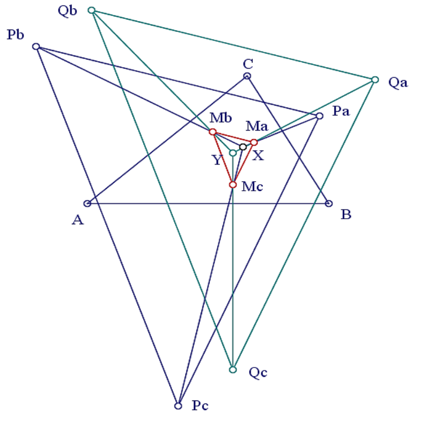

Figure 5

Figure 5 illustrates the construction. In figure 5,
– $PaPbPc$ is the Hatzipolakis triangle of the Retrocenter,
– $QaQbQc$ is the Hatzipolakis triangle of the Malfatti-Moses point,
– X is the Symmedian point, and
– Y is the Malfatti-Moses point.
Then
– lines XPa and YQa concur at point Ma,
– lines XPb and YQb concur at point Mb,
– lines XPc and YQc concur at point Mc, and
– triangle $MaMbMc$ is the Malfatti squares triangle.
We could use e.g. the following constructions:

– the construction of the Hatzipolakis triangle of a point P is given in Ehrmann (Ehrmannq 2002),

– the Retrocenter X(69) could be constructed as the Isotomic conjugate of the Orthocenter,

– the Malfatti-Moses point X(3068) could be constructed as the intersection of the line through the Centroid and the Symmedian point and the line through the Orthocenter and Inner Kenmotu point.

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МАТЕМАТИКА, ОТКРИТА ОТ КОМПЮТРИ: КОНСТРУКЦИИ НА КВАДРАТИТЕ НА МАЛФАТИ

Резюме. Като използваме компютърната програма „Откривател“, авторите намираме 85 различни геометрични конструкции на квадратите на Малфати. От тях 79 конструкции са нови и 6 са известни от по-рано.

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