Educational Technologies
Образователни технологии

POLYNOMIALS OF FOURTH DEGREE WITH COLINEAR CENTRAL SYMMETRIC ROOTS

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Abstract. A geometric relation is considered between a polynomial of fourth degree with symmetrically located roots on a line and the roots of its derivative. A special ellipse is used for the purpose, which is generated by the roots of the polynomial.
Keywords: polynomial of fourth degree; derivative; symmetric points; equilateral triangle; square; center; ellipse; circle; vertices

1. Introduction. It is well known a geometric relation between the roots of a polynomial of fourth degree located in the vertices of a parallelogram and the roots of the corresponding derivative. According to this relation for such a polynomial the roots of its derivative are located in the foci of an ellipse which is tangent to the sides of the parallelogram at their midpoints (Grozdev & Nenkov, 2018 a).

On the other hand, there exist polynomials of fourth degree with roots located in points which are centrally symmetric on a line. For this reason it is possible that geometric relations exist between the roots of the polynomials of this kind and the roots of their derivatives expressed by means of suitable ellipses. It turns out that such relations exist and the corresponding geometric structure differs from the one considered in (Grozdev & Nenkov, 2018 a).

2. A special ellipse and a special circle, determined by two couples of symmetric points on a line. The study of the geometric relations between polynomials of fourth degree with the mentioned properties and their corresponding derivatives will be carried out in a sequence of several preparatory helpful assertions.

Lemma 1. *If the point couples* $\left(A_1, A_3\right)$ *and* $\left(A_2, A_4\right)$ *are located central symmetrically with respect to a point* S *on a line* l*, while the points* P'_{12}, P''_{12}, P'_{23}, P''_{23}, P'_{34}, P''_{34}, P'_{41} *are* P''_{41} *such that the triangles* $A_1A_2P'_{12}$, $A_1A_2P''_{12}$, $A_2A_3P'_{23}$, $A_2A_3P''_{23}$, $A_3A_4P'_{34}$, $A_3A_4P''_{34}$, $A_4A_1P'_{41}$ *and* $A_4A_1P''_{41}$ *are equilateral, then the points* P'_{12}, P''_{12}, P'_{23}, P''_{23}, P'_{34}, P''_{34}, P'_{41} *and* P''_{41} *are located on an ellipse* k *with center* S (Fig. 1).

Poof. Consider a coordinate system with abscissa axis along the line l and the coordinate origin in the center S of symmetry of the point couples $\left(A_1, A_3\right)$ and

(A_2, A_4). With respect to the introduced coordinate system let the abscissae of the points A_1 and A_2 be a_1 and a_2, respectively. It follows from the symmetry with respect to S, that the abscissae of the points A_3 and A_4 are $-a_1$ and $-a_2$, respectively. Use the notation

$$(1) \qquad a_1^2 + a_2^2 = 2R^2 .$$

The coordinates of the vertices of the equilateral triangles could be expressed in the following way:

$$(2) \qquad \begin{aligned}
&P'_{12}\left(\frac{a_1+a_2}{2}, \frac{|a_1-a_2|\sqrt{3}}{2}\right), P''_{12}\left(\frac{a_1+a_2}{2}, -\frac{|a_1-a_2|\sqrt{3}}{2}\right),\\
&P'_{23}\left(\frac{-a_1+a_2}{2}, \frac{|a_1+a_2|\sqrt{3}}{2}\right), P''_{23}\left(\frac{-a_1+a_2}{2}, -\frac{|a_1+a_2|\sqrt{3}}{2}\right),\\
&P'_{34}\left(-\frac{a_1+a_2}{2}, \frac{|a_1-a_2|\sqrt{3}}{2}\right), P''_{34}\left(-\frac{a_1+a_2}{2}, -\frac{|a_1-a_2|\sqrt{3}}{2}\right),\\
&P'_{41}\left(\frac{a_1-a_2}{2}, \frac{|a_1+a_2|\sqrt{3}}{2}\right), P''_{41}\left(\frac{a_1-a_2}{2}, -\frac{|a_1+a_2|\sqrt{3}}{2}\right).
\end{aligned}$$

It follows from (1) and (2), that the eight points under consideration are located on the ellipse k, which is determined by the equation

$$(3) \qquad k: \frac{x^2}{R^2} + \frac{y^2}{3R^2} = 1 .$$

Thus, the lemma is proven.

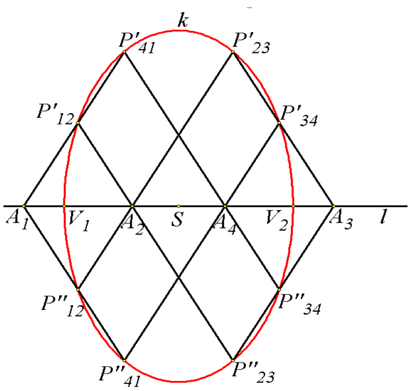

Fig. 1

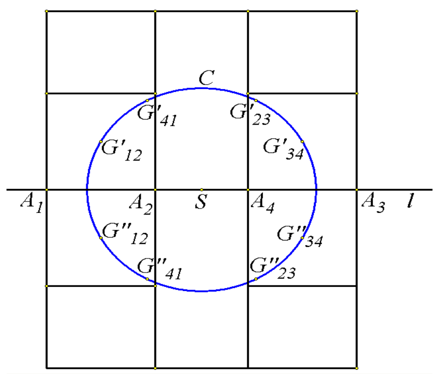

Fig. 2

Lemma 2. *If the point couples* (A_1, A_3) *and* (A_2, A_4) *are located central symmetrically with respect to a point* S *on a line* l, *while the point couples* (G'_{12}, G''_{12}), (G'_{23}, G''_{23}), (G'_{34}, G''_{34}) *and* (G'_{41}, G''_{41}) *are the centers of the squares with bases* A_1A_2, A_2A_3, A_3A_4 *and* A_4A_1*, respectively, then the points* G'_{12}, G''_{12}, G'_{23}, G''_{23}, G'_{34}, G''_{34}, G'_{41} *and* G''_{41} *are located on a circle C with center* S (Fig. 2).

Proof. Consider the same coordinate system as in the proof of lemma 1. The coordinates of the point under consideration could be expressed in the following way:

$$(4)\qquad \begin{aligned} &G'_{12}\left(\frac{a_1+a_2}{2}, \frac{|a_1-a_2|}{2}\right), G''_{12}\left(\frac{a_1+a_2}{2}, -\frac{|a_1-a_2|}{2}\right),\\ &G'_{23}\left(\frac{-a_1+a_2}{2}, \frac{|a_1+a_2|}{2}\right), G''_{23}\left(\frac{-a_1+a_2}{2}, -\frac{|a_1+a_2|}{2}\right),\\ &G'_{34}\left(-\frac{a_1+a_2}{2}, \frac{|a_1-a_2|}{2}\right), G''_{34}\left(-\frac{a_1+a_2}{2}, -\frac{|a_1-a_2|}{2}\right),\\ &G'_{41}\left(\frac{a_1-a_2}{2}, \frac{|a_1+a_2|}{2}\right), G''_{41}\left(\frac{a_1-a_2}{2}, -\frac{|a_1+a_2|}{2}\right). \end{aligned}$$

It follows easily from (1) and (4), that the eight points are located on the circle C, determined by the equation

$$(5)\qquad C : x^2 + y^2 = R^2 .$$

This ends the proof of the lemma.

It follows from the equations (3) and (5), that the only common points of k and C are the vertices $V_1(-R, 0)$ and $V_2(R, 0)$ of k.

Thus, we have proved also the following:

Lemma 3. *The circle* C *is tangent to the ellipse* k *at the vertices* V_1 *and* V_2 *of the small axis of* k (Fig. 3).

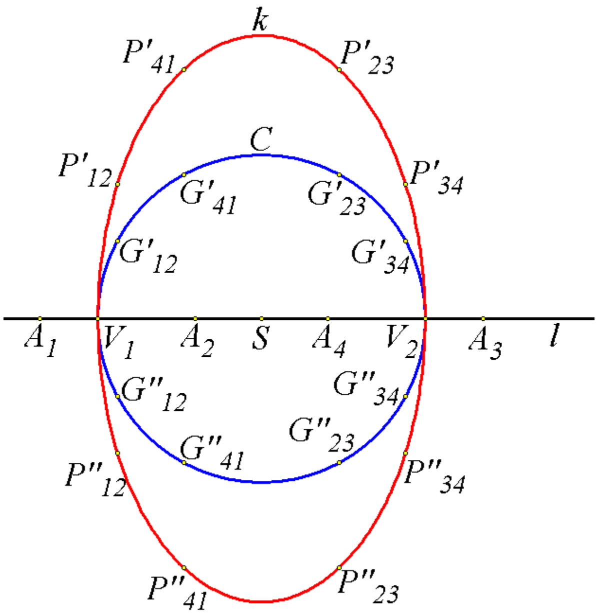

Fig. 3

3. A relation between the roots of the derivative of a polynomial of fourth degree with central symmetric colinear roots and the vertices of the small axis and the center of a special ellipse. We will show now, that the vertices V_1 and V_2 of k define the geometric relation we are looking for between the polynomials that were mentioned at the beginning and their derivatives. More precisely, the following theorem is true:

Theorem. *If a polynomial $P(z)$ of fourth degree of a complex variable and with complex coefficients has roots in the point couples (A_1, A_3) and (A_2, A_4), which are located central symmetrically with respect to a point S on a line l, then the derivative $P'(z)$ of $P(z)$ has roots in the vertices on the small axis and the center S of the ellipse k, determined by the vertices of the equilateral triangles, which are constructed on the segments A_1A_2, A_2A_3, A_3A_4 and A_4A_1* (Fig. 4, 5).

Proof. Consider the Gauss coordinate system with a real axis along the line l. This means, that it is possible to use the coordinate system and the notations from the proof of lemma 1. Let now $P(z)$ be a standardized polynomial of fourth degree with roots in the points A_1, A_2, A_3 and A_4. It follows from Vieta's formulae, that $P(z)$ and its derivative $P'(z)$ could be represented in the following way:

$$P(z) = z^4 - 2R^2 z^2 + a_1^2 a_2^2, \tag{6}$$

$$P'(z) = 4z^3 - 4R^2 z. \tag{7}$$

It follows from (7), that the roots of $P'(z)$ are $z_1 = -R$, $z_2 = 0$ and $z_3 = R$, which proves the theorem for a standardized polynomial. Since each polynomial of fourth degree with the mentioned properties could be reduced to a similar form, then the theorem turns out to be true for all polynomials of the kind under consideration.

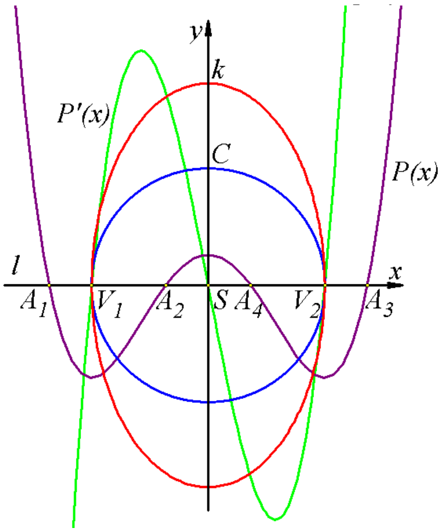

Fig. 4

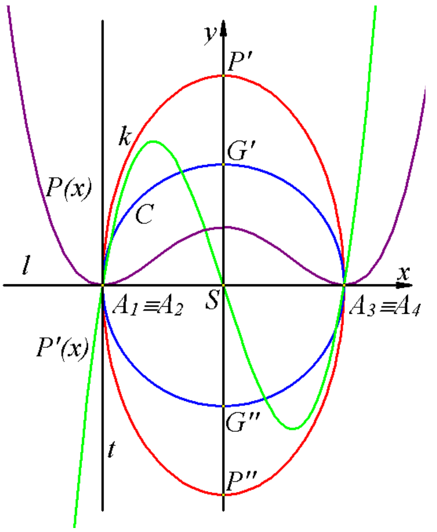

Fig.5

The proven theorem is reasonable in the case $A_1 \equiv A_2$ too. Then $A_3 \equiv A_4$, $P'_{12} \equiv P'_{23} \equiv P'_{34} \equiv P'_{41} \equiv P'$, $P''_{12} \equiv P''_{23} \equiv P''_{34} \equiv P''_{41} \equiv P''$, $G'_{12} \equiv G'_{23} \equiv G'_{34} \equiv G'_{41} \equiv G'$, $G''_{12} \equiv G''_{23} \equiv G''_{34} \equiv G''_{41} \equiv G''$. The circle C is fully determined by its center S and the point A_1 belonging to it. The ellipse k is fully determined by the four points A_1, A_3, P', P'' and the tangent t through A_1 (or A_3), which is perpendicular to l. In this case the vertices of the ellipse on its small axis coincide with the points A_1 and A_3, while the vertices on the big axis are P' and P''. In fact, the polynomial $P(z)$ has double roots in A_1 and A_3, which means that $P'(z)$ has simple roots in the same two points. The cases when $P(z) = P(x)$ is a polynomial with real coefficients of the real variable x, are demonstrated in Fig. 4 and Fig. 5.

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ПОЛИНОМИ ОТ ЧЕТВЪРТА СТЕПЕН С КОЛИНЕАРНИ ЦЕНТРАЛНО СИМЕТРИЧНИ КОРЕНИ

Резюме. Разгледана е геометрична връзка между полином от четвърта степен с корени, разположени симетрично върху права, и корените на неговата производна. За целта е използвана специална елипса, породена от корените на полинома.

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