Educational Technologies
Образователни технологии

PROBLEM 6. FROM IMO'2018

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Abstract. The International Mathematical Olympiad is one of the respectable events and one of the most long-lived international educational and scientific competitions. It is the largest, oldest and most prestigious scientific Olympiad for high school students. The 59th edition of the event took place in Cluj-Napoca, Romania, 3 – 14 July 2018. The present paper is dedicated to the sixth problem on the Olympiad paper. A detailed analysis of the problem is proposed in a methodological way, which will be useful for students and teachers in the preparatory process for future participations in mathematical competitions.

Keywords: Olympiad; problem solving

The problem 6 on the paper of the 59th International mathematical Olympiad was solved fully (7 points) by 18 participants, 5 students were marked with 6 points, 2 with 5 points, 5 with 4 points, 11 with 3 points, 26 with 2 points, 108 with 1 point and 419 with 0 points. The mean result of all the 594 participants in the Olympiad from 107 countries is 0. 638, which shows that the problem is hard and needs a detailed analysis.

Problem 6. A convex quadrilateral *ABCD* satisfies $AB.CD = BC.DA$. Point X lies inside *ABCD* so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle BXA + \angle DXC = 180°$.

Lemma 1. Each convex quadrilateral has a unique interior point X such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

Proof: The following cases are possible:

1) If $ABCD$ is a parallelogram, then the point X is the intersection point of the diagonals. This follows from the equality of the cross-opposite angles of the parallelogram. Reversely, the equality of the mentioned angles implies that the point X should belong to each of the diagonals and consequently it belongs to both the diagonals simultaneously. Thus the point X is unique.

2.1) If $ABCD$ is a trapezoid with $AB \parallel CD$ and $BC \cap AD = W$, then the common point of the diagonal AC and the circumcircle $k(BDW)$ of ΔBDW is the desired point X. The proof is the following:

$$\text{i)}\ \angle XAB = \angle XCD \text{ (cross-opposite angles);}$$

$$\text{ii)}\ \angle XBC = \frac{\overset{\frown}{XDW}}{2} = \frac{\overset{\frown}{XD} + \overset{\frown}{DW}}{2} = \angle XDA.$$

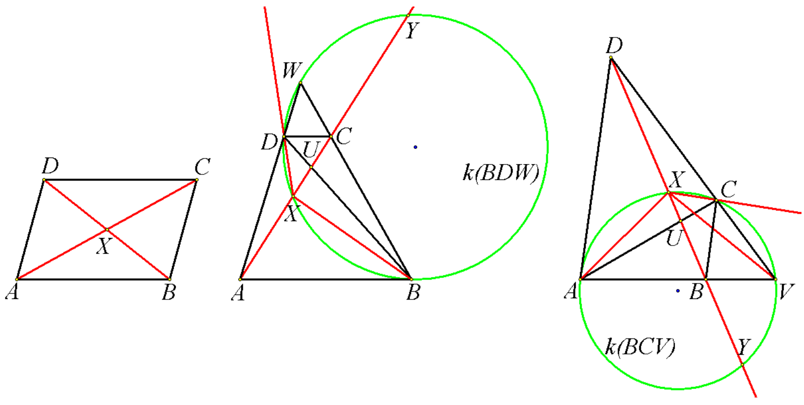
Reversely, the equality $\angle XAB = \angle XCD$ implies, that the point X belongs to the diagonal AC, while the equality $\angle XBC = \angle XDA$ is verified when X is on the circle $k(BDW)$. Since AC and $k(BDW)$ have only one common point, which is in the interior of $ABCD$, then the point X is unique. The second intersection point Y of the line AC and $k(BDW)$ is such, that $\angle YAB = 180° - \angle YCD$ and $\angle YBC = 180° - \angle YDA$.

2.2) If $ABCD$ is a trapezoid with $BC \parallel AD$ and $AB \cap CD = V$, then the common point of the diagonal BD and the circumcircle $k(ACV)$ of ΔACV is the desired point X. The proof is the following:

$$\text{i)}\ \angle XAB = \frac{\overset{\frown}{XCV}}{2} = \frac{\overset{\frown}{XC} + \overset{\frown}{CV}}{2} = \angle XCD;$$

$$\text{ii)}\ \angle XBC = \angle XDA \text{ (cross-opposite angles).}$$

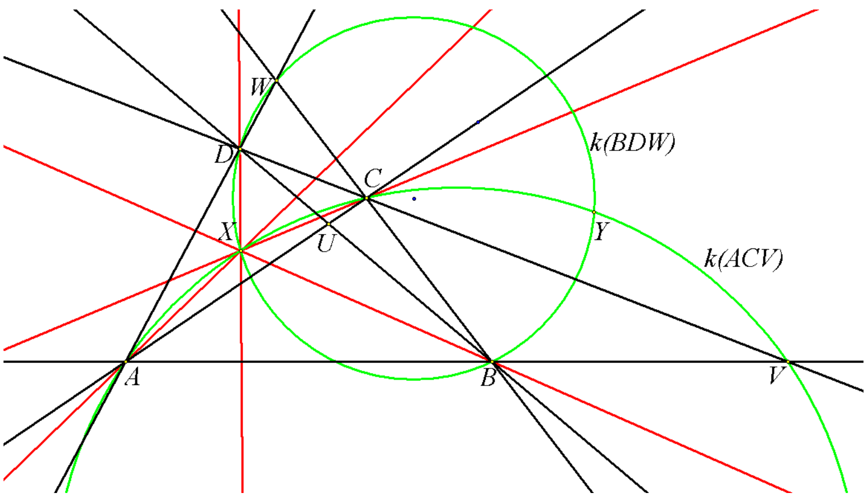
The uniqueness of the point X could be proved in the same manner as in the previous case. The second common point Y of the line BD and $k(ACV)$ is such, that $\angle YAB = 180° - \angle YCD$ and $\angle YBC = 180° - \angle YDA$.



3) If $ABCD$ is without parallel sides and $AB \cap CD = V$, $BC \cap AD = W$, then the interior point of $ABCD$, which is the intersection point of the circumcircles $k(ACV)$ and $k(BDW)$ of ΔACV and ΔBDW, respectively, is the desired point X. The proof is the following:

$$\text{i) } \angle XAB = \frac{\overset{\frown}{XCV}}{2} = \frac{\overset{\frown}{XC} + \overset{\frown}{CV}}{2} = \angle XCD;$$

$$\text{ii) } \angle XBC = \frac{\overset{\frown}{XDW}}{2} = \frac{\overset{\frown}{XD} + \overset{\frown}{DW}}{2} = \angle XDA.$$



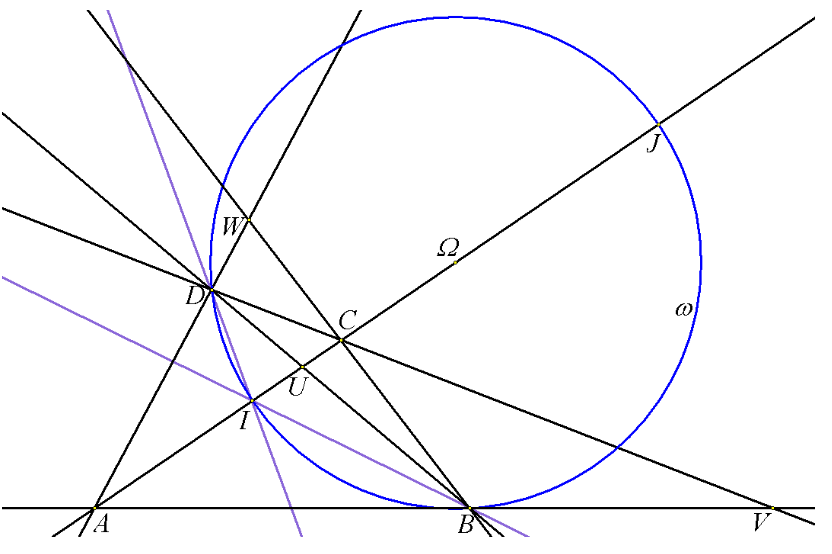
Reversely, the equality $\angle XAB = \angle XCD$ means, that the point X belongs to the circle $k(ACV)$, while the equality $\angle XBC = \angle XDA$ is verified when X is on the circle $k(BDW)$. Since AC and $k(BDW)$ have only one common point, which is in the interior of $ABCD$, then the point X is unique. The second intersection point Y of $k(ACV)$ and $k(BDW)$ is such, that $\angle YAB = 180° - \angle YCD$ and $\angle YBC = 180° - \angle YDA$.

Lemma 2. If the sides of the convex quadrilateral $ABCD$ satisfy the equality $AB.CD = BC.DA$, then:

a) the angular bisectors of the angles ABC and CDA meet in the point I from the diagonal AC;

b) the angular bisectors of the angles BAD and DCB meet in the point I' from the diagonal BD.

Proof: Rewrite the given equality in the form $\frac{BA}{BC} = \frac{DA}{DC}$. If I is such a point on the diagonal AC, that $\frac{IA}{IC} = \frac{BA}{BC}$, then BI is the angular bisector of $\angle ABC$. On the other hand, the equality $\frac{IA}{IC} = \frac{DA}{DC}$ is true, which means, that DI is the angular bisector of $\angle CDA$. The assertion b) could be obtained analogously.

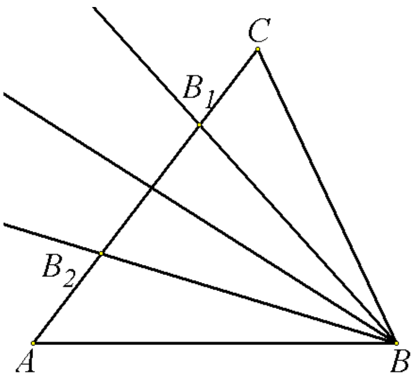


In fact, if $BA \neq BC$, then there exists a point J on the line BC with the property $\frac{JA}{JC} = \frac{BA}{BC}$. The circle ω with diameter IJ is the locus of the points M, for which $\frac{MA}{MC} = \frac{BA}{BC}$. The circle ω is known to be the *Apollonius circle* for the segment AC under the ratio $\frac{BA}{BC}$. For this reason, if an arbitrary ΔABC is given, then the point D from the Apollonius circle ω is the fourth vertex of the quadrilateral $ABCD$, for which $AB.CD = BC.DA$. In such a way we come upon an idea for the construction of a quadrilateral satisfying the conditions of the problem under consideration. On the other hand, lemma 1 gives a possibility to construct the point X for this quadrilateral.

Lemma 3. Let ABC be an arbitrary triangle, while B_1 and B_2 be such points on the line AC, that the lines BB_1 and BB_2 are symmetric with respect to the angular bisector of $\angle ABC$. Then $\frac{\overline{CB_1}}{\overline{AB_1}}.\frac{\overline{CB_2}}{\overline{AB_2}} = \frac{BC^2}{AB^2}$.

Proof: For the areas of the triangles BCB_1 and BAB_2 we have, that $\frac{S_{BCB_1}}{S_{BAB_2}} = \frac{CB_1}{AB_2}$ and $\frac{S_{BCB_1}}{S_{BAB_2}} = \frac{BC.BB_1.\sin\angle B_1BC}{AB.BB_2.\sin\angle B_2BA} = \frac{BC.BB_1}{AB.BB_2}$. Consequently $\frac{CB_1}{AB_2} = \frac{BC.BB_1}{AB.BB_2}$.

Analogously, using the areas of the triangles CBB_2 and ABB_1 we get the equality $\frac{CB_2}{AB_1} = \frac{BC.BB_2}{AB.BB_1}$. Multiplying the obtained equalities we find, that $\frac{CB_1}{AB_1}.\frac{CB_2}{AB_2} = \frac{BC^2}{AB^2}$.

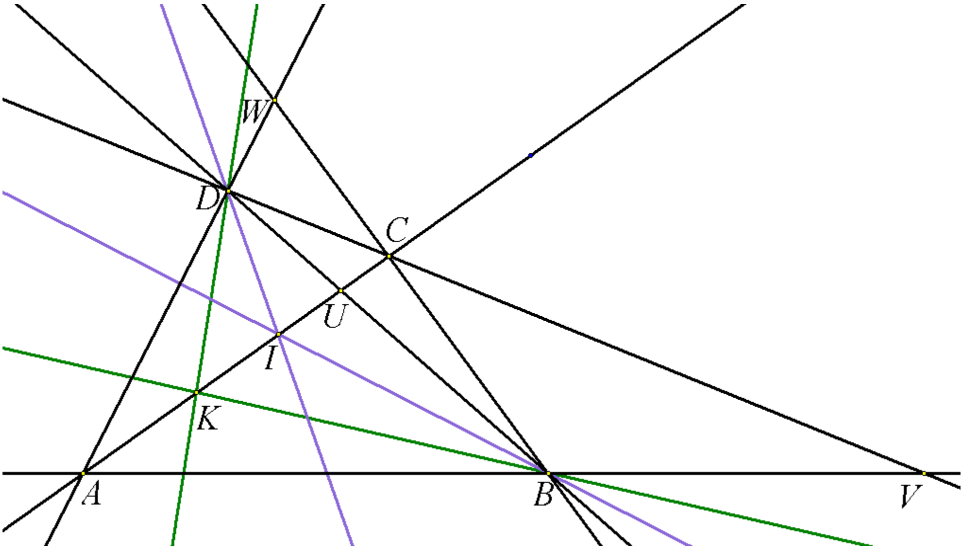


Lemma 4. If the convex quadrilateral $ABCD$ satisfies the equality $AB.CD = BC.DA$, then:

a) the symmetric images of the diagonal BD with respect to the angular bisectors of $\angle ABC$ and $\angle CDA$ meet in a point K from the diagonal AC;

b) the symmetric images of the diagonal AC with respect to the angular bisectors of $\angle BAD$ and $\angle DCB$ meet in a point K' from the diagonal BD.

Proof: Denote by U the common point of the diagonals AC and BD. Let the symmetric images of BD with respect to the angular bisectors of $\angle ABC$ and $\angle CDA$ meet AC in the points K_1 and K_2, respectively. Applying lemma 3 to ΔABC we obtain the equality $\frac{CU}{AU}.\frac{CK_1}{AK_1} = \frac{BC^2}{AB^2}$. Applying lemma 3 to ΔCDA we obtain $\frac{AU}{CU}.\frac{AK_2}{CK_2} = \frac{DA^2}{CD^2}$. Multiplying the two equalities we deduce, that $\frac{CK_1.AK_2}{AK_1.CK_2} = \left(\frac{BC.DA}{AB.CD}\right)^2 = 1$. Therefore, $\frac{AK_2}{CK_2} = \frac{AK_1}{CK_1}$. Since the points K_1 and K_2 belong to the segment AC, the last equality means, that the points K_1 and K_2 coincide. The assertion for the point K' could be proven analogously.



It follows from lemma 4 that the point I is the center of the incircle of ΔBDK. Thus, AC is the angular bisector of $\angle BKD$. Consequently $180^\circ = \angle AKD + \angle DKC = \angle AKD + \angle BKC$. Thus we obtain the following

Conclusion. The equalities $\angle AKD + \angle BKC = 180^\circ$ and $\angle BK'A + \angle CK'D = 180^\circ$ are satisfied.

Lemma 5. If the convex quadrilateral $ABCD$ satisfies the equality $AB.CD = BC.DA$, then:

a) The second common point X of the circumcircles $k(BCK)$ and $k(DAK)$ of ΔBCK and ΔDAK, respectively, is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

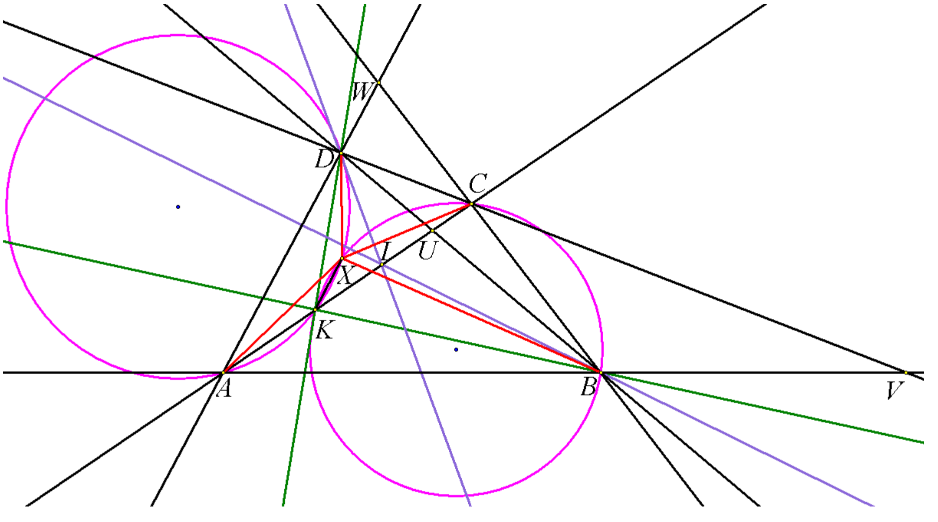
b) The second common point X of the circumcircles $k(BCK')$ and $k(DAK')$ of $\Delta BCK'$ and $\Delta DAK'$, respectively, is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

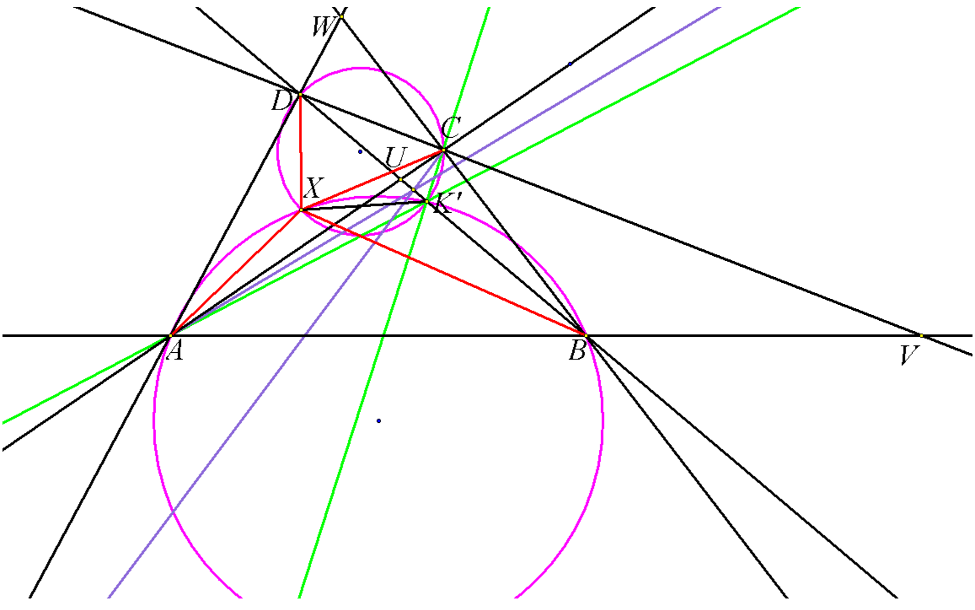
Proof: We will consider the first case only. Let $AB \cap AD = V$ and the second common point of $k(BCK)$ and $k(DAK)$ be X. In order to prove the first equality it is enough to establish, that the point X belongs to the circumcircle $k(ACV)$ of ΔACV (as shown in lemma 1). The following equalities are true:

$$\angle CXA = \angle CXK + \angle KXA = 180^\circ - \angle CBK + \frac{\overset{\frown}{AK}}{2} = 180^\circ - \angle DBA + \angle ADK =$$
$$= \angle DBV + \angle BDV = 180^\circ - \angle AVD .$$

Consequaently $\angle CXA + \angle AVD = 180^\circ$. This equality means that the point X belongs to the circumcircle of ΔACV . For this reason $\angle XAB = \angle XCD$. The second equality could be obtained in the following way: $\angle XBC = \frac{\overset{\frown}{XC}}{2} = \angle XKC = 180^\circ - \angle AKX = \angle XDA$.

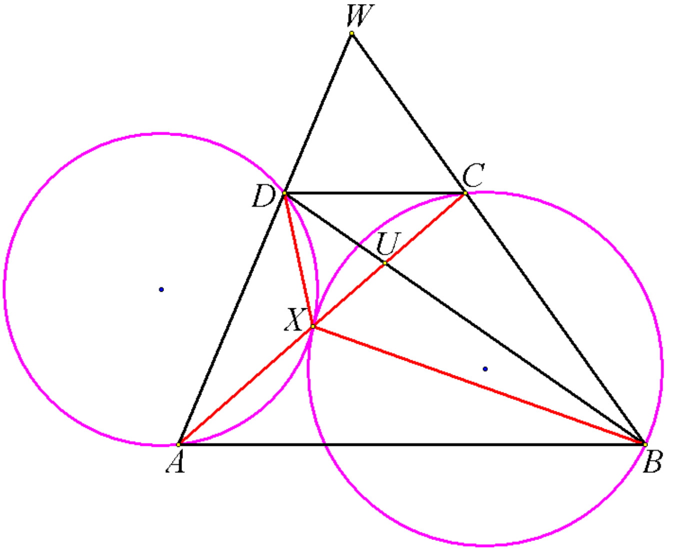
The assertion for the other couple of circles under the condition $AD \cap BC = W$ could be proven in the same way.





If $AB \parallel CD$ and the second common point of $k(BCK)$ and $k(DAK)$ is X, as in the case just considered, we obtain $\angle CXA = 180° - \angle DBA + \angle BDC = 180°$. This means that the points C, X and A are collinear, i.e. the point X is on the diagonal AC. Consequently, the common points X and K of the circles $k(BCK)$ and $k(DAK)$ are on the diagonal AC. But the circle $k(BCK)$ intersects the line AC in the point C. Since a circle and a line could have no more than two common points, then the point X coincides with K. Therefore, in this case the circles $k(BCK)$ and $k(DAK)$ are tangent at the point X.

Analogously, if $AD \parallel BC$, the circles $k(BCK')$ and $k(DAK')$ are tangent at the point $X \equiv K'$.



Solution of problem 6: If $ABCD$ is a parallelogram, then the equality $AB.CD = BC.DA$ is satisfied only in the case when $ABCD$ is a rhombus. Since the point X is the intersection point of its diagonals, then $\angle BXA + \angle DXC = 90° + 90° = 180°$. Let now $ABCD$ is such, that two of its sides at least are not parallel to each other. For definiteness let $AB \cap CD = V$. Since (according to lemma 5) the point X belongs to the circles $k(BCK)$ and $k(DAK)$, then $\angle DXA + \angle BXC = \angle AKD + \angle BKC$. From the conclusion we obtain, that $\angle DXA + \angle BXC = 180°$. But $(\angle BXA + \angle DXC) + (\angle DXA + \angle BXC) = 360°$. Therefore $\angle BXA + \angle DXC = 180°$. If we consider the point X as a common point of the circles $k(BCK')$ and $k(DAK')$, we deduce the desired equality as a consequence of lemma 5 and the conclusion in the following way:

$$\angle BXA + \angle DXC = \angle BK'A + \angle CK'D = 180°.$$

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ЗАДАЧА 6. ОТ IMO'2018

Резюме. Международната олимпиада по математика е едно от уважаваните събития и едно от най-старите международни образователни и научни състезания. Тя е най-голямата, най-старата и най-престижната научна олимпиада за гимназиални ученици. 59-ото издание на събитието се проведе в Клуж-Напока, Румъния, в периода 3 – 14 юли 2018. Настоящата статия е посветена на шестата задача от темата. Предложен е подробен анализ на задачата с методологически характер, който ще бъде полезен за ученици и учители по време на подготовката за бъдещи участия в математически състезания.

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