

THE REARRANGEMENT INEQUALITY

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Abstract. In this paper we consider a really very useful inequality, the so called rearrangement inequality, which has many applications and could be used in proving other inequalities. The paper contains a proof of the rearrangement inequality and several examples of its application.

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Theorem 1. Consider two collections of real numbers in increasing order:

$$a_1 \leq a_2 \leq \dots \leq a_n \text{ and } b_1 \leq b_2 \leq \dots \leq b_n.$$

For any permutation $(a'_1, a'_2, \dots, a'_n)$ of $(a_1, a_2, \dots, a_n)$, it happens that

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a'_1 b_1 + a'_2 b_2 + \dots + a'_n b_n \quad (1)$$

$$\geq a_n b_1 + a_{n-1} b_2 + \dots + a_1 b_n. \quad (2)$$

Moreover, the equality in (1) holds true iff $(a'_1, a'_2, \dots, a'_n) = (a_1, a_2, \dots, a_n)$ and the equality in (2) holds true iff $(a'_1, a'_2, \dots, a'_n) = (a_n, a_{n-1}, \dots, a_1)$. (1) is known to be the **rearrangement inequality**.

Proof of the rearrangement inequality: Suppose that $b_1 \leq b_2 \leq \dots \leq b_n$. Let

$$S = a_1 b_1 + a_2 b_2 + \dots + a_r b_r + \dots + a_s b_s + \dots + a_n b_n,$$

$$S' = a_1 b_1 + a_2 b_2 + \dots + a_s b_r + \dots + a_r b_s + \dots + a_n b_n.$$

The difference between S and S' is that the coefficients of b_r and b_s , where $r < s$, are switched. Hence,

$$S - S' = a_r b_r + a_s b_s - a_s b_r - a_r b_s = (b_s - b_r)(a_s - a_r).$$

Thus, we have that $S \geq S'$ iff $a_s \geq a_r$. Repeating this process we get as a result that the sum S is maximal when $a_1 \leq a_2 \leq \dots \leq a_n$.

Corollary 1. For any permutation $(a'_1, a'_2, \dots, a'_n)$ of $(a_1, a_2, \dots, a_n)$, we have that

$$a_1^2 + a_2^2 + \dots + a_n^2 \geq a_1 a'_1 + a_2 a'_2 + \dots + a_n a'_n.$$

Corollary 2. For any permutation $(a'_1, a'_2, \dots, a'_n)$ of $(a_1, a_2, \dots, a_n)$, we have that

$$\frac{a'_1}{a_1} + \frac{a'_2}{a_2} + \dots + \frac{a'_n}{a_n} \geq n.$$

In the sequel we propose several examples of application of the rearrangement inequality.

Example 1. Let a, b and c be positive real numbers. Prove **Nesbitt's inequality**

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Solution: Without loss of generality, we may assume that $a \geq b \geq c$. Then clearly

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

By the rearrangement inequality we deduce

$$\begin{aligned} \text{and} \quad & \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b} \\ & \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}. \end{aligned}$$

Adding the last two inequalities we obtain that

$$\begin{aligned} \text{or} \quad & 2 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \geq 3 \\ & \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}, \text{ q.e.d.} \end{aligned}$$

Equality holds true iff $a=b=c$.

Example 2. (IMO 1978.) Let $x_1, x_2, \dots, x_n$ be different positive integers. Prove the inequality

$$\frac{x_1}{1^2} + \frac{x_2}{2^2} + \dots + \frac{x_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}.$$

Solution: Let $(a_1, a_2, \dots, a_n)$ be a permutation of $(x_1, x_2, \dots, x_n)$ with $a_1 \leq a_2 \leq \dots \leq a_n$ and $(b_1, b_2, \dots, b_n) = \left(\frac{1}{n^2}, \frac{1}{(n-1)^2}, \dots, \frac{1}{1^2} \right)$, that is $b_i = \frac{1}{(n+1-i)^2}$ for $i=1, 2, \dots, n$.

Consider the permutation $(a'_1, a'_2, \dots, a'_n)$ of $(a_1, a_2, \dots, a_n)$ defined by $a'_i = x_{n+1-i}$, for $i=1, 2, \dots, n$. Using inequality (2) we may claim that

$$\begin{aligned} \frac{x_1}{1^2} + \frac{x_2}{2^2} + \dots + \frac{x_n}{n^2} &= a'_1 b_1 + a'_2 b_2 + \dots + a'_n b_n \\ &\geq a_n b_1 + a_{n-1} b_2 + \dots + a_1 b_n \\ &= a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1 \\ &= \frac{a_1}{1^2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2}. \end{aligned}$$

Since $1 \leq a_1, 2 \leq a_2, \dots, n \leq a_n$ we have that

$$\frac{x_1}{1^2} + \frac{x_2}{2^2} + \dots + \frac{x_n}{n^2} \geq \frac{a_1}{1^2} + \frac{a_2}{2^2} + \dots + \frac{a_n}{n^2} \geq \frac{1}{1^2} + \frac{2}{2^2} + \dots + \frac{n}{n^2} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}, \text{ q.e.d.}$$

Equality holds true iff $x_1=1, x_2=2, \dots, x_n=n$.

Example 3. Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a^2 + c^2}{b} + \frac{b^2 + a^2}{c} + \frac{c^2 + b^2}{a} \geq 2(a+b+c).$$

Solution: Since the given inequality is symmetric, without loss of generality we may assume that $a \geq b \geq c$. Then clearly $a^2 \geq b^2 \geq c^2$ and $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$.

By the rearrangement inequality we have

$$\begin{aligned} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} &= a^2 \cdot \frac{1}{b} + b^2 \cdot \frac{1}{c} + c^2 \cdot \frac{1}{a} \geq a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} = a+b+c, \quad (3) \\ \text{and} \quad \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} &= a^2 \cdot \frac{1}{c} + b^2 \cdot \frac{1}{a} + c^2 \cdot \frac{1}{b} \geq a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} = a+b+c. \quad (4) \end{aligned}$$

Adding (3) and (4) yields the required inequality. The equality occurs iff $a=b=c$.

Example 4. Let x, y, z be positive real numbers. Prove the inequality

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z.$$

Solution: Since the given inequality is symmetric we may assume that $x \geq y \geq z$. Then

$$x^3 \geq y^3 \geq z^3 \quad \text{and} \quad \frac{1}{yz} \geq \frac{1}{zx} \geq \frac{1}{xy}.$$

By the rearrangement inequality we have

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} = x^3 \cdot \frac{1}{yz} + y^3 \cdot \frac{1}{zx} + z^3 \cdot \frac{1}{xy} \geq x^3 \cdot \frac{1}{xy} + y^3 \cdot \frac{1}{yz} + z^3 \cdot \frac{1}{zx} = \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x}. \quad (5)$$

We will prove that

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z. \quad (6)$$

Let $x \geq y \geq z$. Then $x^2 \geq y^2 \geq z^2$ and $\frac{1}{z} \geq \frac{1}{y} \geq \frac{1}{x}$ (since inequality (6) is cyclic, we need to consider the case $z \geq y \geq x$).

By the rearrangement inequality we obtain

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq \frac{x^2}{x} + \frac{y^2}{y} + \frac{z^2}{z} = x + y + z.$$

The case when $z \geq y \geq x$ is analogous to the previous one.

Now by (5) and (6) we obtain

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geq x + y + z, \text{ q.e.d.}$$

Equality occurs iff $x = y = z$.

Example 5. (IMO 1964). Suppose that a, b, c are the lengths of the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c) \leq 3abc.$$

Solution: Since the expression is a symmetric function of a, b and c , we can assume, without loss of generality, that $c \leq b \leq a$. In this case, $a(b+c-a) \leq b(a+c-b) \leq c(a+b-c)$. For instance, the first inequality could be proved in the following way:

$$a(b+c-a) \leq b(a+c-b) \Leftrightarrow ab+ac-a^2 \leq ab+bc-b^2$$

$$\Leftrightarrow (a-b)c \leq (a+b)(a-b)$$

$$\Leftrightarrow (a-b)(a+b-c) \geq 0.$$

By (2) of the rearrangement inequality, we have

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq bc(b+c-a) + cb(c+a-b) + ac(a+b-c),$$

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq ca(b+c-a) + ab(c+a-b) + bc(a+b-c).$$

Therefore, it follows after summing these inequalities, that:

$$2[a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)] \leq 6abc,$$

i.e.

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc, \text{ q.e.d.}$$

Equality holds iff $a=b=c$ (equilateral triangle).

Example 6. (IMO 1983). Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Solution: Consider the case $c \leq b \leq a$ (the other cases are similar). As in the previous example, we have that

$$a(b+c-a) \leq b(a+c-b) \leq c(a+b-c)$$

and since $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$, using the inequality (1), we deduce, that:

$$\frac{1}{a} \cdot a(b+c-a) + \frac{1}{b} \cdot b(c+a-b) + \frac{1}{c} \cdot c(a+b-c) \geq \frac{1}{c} \cdot a(b+c-a) + \frac{1}{a} \cdot b(c+a-b) + \frac{1}{b} \cdot c(a+b-c).$$

Therefore,

$$a+b+c \geq \frac{a(b-a)}{c} + \frac{b(c-b)}{a} + \frac{c(a-c)}{b} + a+b+c.$$

It follows that

$$\frac{a(b-a)}{c} + \frac{b(c-b)}{a} + \frac{c(a-c)}{b} \leq 0.$$

Multiplying by abc , we obtain

$$a^2b(b-a) + b^2c(b-c) + c^2a(c-a) \geq 0, \text{ q.e.d.}$$

Remark 1. Four different proofs of this inequality could be found in (Arslanagić, 2005).

Example 7. Let $a_1, a_2, \dots, a_n$ be distinct positive integers. Show that

$$\frac{a_1}{2} + \frac{a_2}{8} + \dots + \frac{a_n}{n \cdot 2^n} \geq 1 - \frac{1}{2^n}.$$

Solution: Arrange $a_1, a_2, \dots, a_n$ in increasing order as $b_1, b_2, \dots, b_n$. Then $b_m \geq m$ because we have distinct positive integers. Since $\frac{1}{2}, \frac{1}{8}, \dots, \frac{1}{n \cdot 2^n}$, by the rearrangement inequality it follows that:

$$\begin{aligned} \frac{a_1}{2} + \frac{a_2}{8} + \dots + \frac{a_n}{n \cdot 2^n} &\geq \frac{b_1}{2} + \frac{b_2}{8} + \dots + \frac{b_n}{n \cdot 2^n} \\ &\geq \frac{1}{2} + \frac{2}{8} + \dots + \frac{n}{n \cdot 2^n} \\ &= \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} \\ &= \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} \right) \\ &= \frac{1}{2} \cdot \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}, \text{ q.e.d.} \end{aligned}$$

Equality holds true iff $a_1 = 1, a_2 = 2, \dots, a_n = n$.

Example 8. (West German Math Olympiad, 1982). If $a_1, a_2, \dots, a_n > 0$ and $a = a_1 + a_2 + \dots + a_n$; then

$$\sum_{i=1}^n \frac{a_i}{2a - a_i} \geq \frac{n}{2n-1}.$$

Solution: By symmetry we may assume that $a_1 \geq a_2 \geq \dots \geq a_n$. Then

$$\frac{1}{2a - a_n} \leq \dots \leq \frac{1}{2a - a_1}.$$

For convenience let $a_i = a_j$ if $i \equiv j \pmod{n}$. For $m = 0, 1, \dots, n-1$ by the rearrangement inequality we get

$$\sum_{i=1}^n \frac{a_{m+i}}{2a - a_i} \leq \sum_{i=1}^n \frac{a_i}{2a - a_i}.$$

Adding these n inequalities we have

$$\sum_{i=1}^n \frac{a}{2a-a_i} \leq \sum_{i=1}^n \frac{na_i}{2a-a_i}.$$

Since

$$\frac{a}{2a-a_i} = \frac{1}{2} + \frac{1}{2} \cdot \frac{a_i}{2a-a_i},$$

we get

$$\frac{n}{2} + \frac{1}{2} \sum_{i=1}^n \frac{a_i}{2a-a_i} \leq n \sum_{i=1}^n \frac{a_i}{2a-a_i}.$$

From here we obtain the desired inequality.

Equality holds iff $a_1 = a_2 = \dots = a_n = 1$.

Example 9. If $a, b, c > 0$, prove tha

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq \frac{a^2+b^2+c^2}{2}.$$

Solution: By symmetry we may assume that $a \leq b \leq c$. Then $a+b \leq c+a \leq b+c$. So, we have

$$\frac{1}{b+c} \leq \frac{1}{c+a} \leq \frac{1}{a+b}.$$

By the rearrangement inequality we have

$$\begin{aligned} \frac{a^3}{a+b} + \frac{b^3}{b+c} + \frac{c^3}{c+a} &\leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}, \\ \frac{a^3}{c+a} + \frac{b^3}{a+b} + \frac{c^3}{b+c} &\leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}. \end{aligned}$$

Adding these inequalities and then dividing by 2, we get

$$\frac{1}{2} \left(\frac{a^3+b^3}{a+b} + \frac{b^3+c^3}{b+c} + \frac{c^3+a^3}{c+a} \right) \leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}.$$

Finally, since

$$\frac{x^3+y^3}{x+y} = x^2 - xy + y^2 \geq \frac{x^2+y^2}{2},$$

we have

$$\frac{a^2+b^2+c^2}{2} = \frac{1}{2} \left(\frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} \right) \leq \frac{1}{2} \left(\frac{a^3+b^3}{a+b} + \frac{b^3+c^3}{b+c} + \frac{c^3+a^3}{c+a} \right) \leq \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}, \text{ i.e.}$$

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq \frac{a^2+b^2+c^2}{2}, \text{ q.e.d.}$$

Equality holds true iff $a=b=c$.

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