

## KLAMKIN'S INEQUALITY AND ITS APPLICATION

Šefket Arslanagić, Daniela Zubović  
*University of Sarajevo (Bosnia and Herzegovina)*

**Abstract.** In this paper we consider a very useful inequality that Murray Klamkin<sup>1)</sup> proved in 1975 (Uldmkin, 1975). The inequality has many applications, proving new inequalities included. A proof and some applications are proposed.

*Keywords:* Klamkin's inequality; triangle sides; scalar product; law of cosines; application; examples

**Theorem 1. (Klamkin's inequality).** Let  $x, y$  and  $z$  be real numbers such that  $x+y+z>0$ . Then for any point  $P$  in the plane of the triangle  $\triangle ABC$  the following inequality holds true:

$$x|PA|^2 + y|PB|^2 + z|PC|^2 \geq \frac{yza^2 + zxb^2 + xyc^2}{x+y+z}, \quad (1)$$

where  $a, b, c$  are the lengths of the sides of the triangle  $\triangle ABC$ .

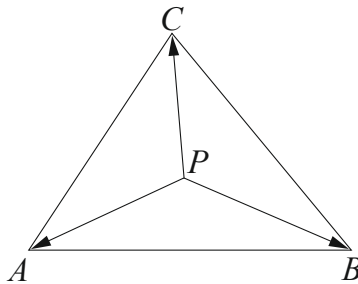
The equality in (1) holds if and only if the point  $P$  satisfies the equality

$$\vec{AP} = \frac{y}{x+y+z} \vec{AB} + \frac{z}{x+y+z} \vec{AC}. \quad (2)$$

**Proof:** Observe the vector  $x\vec{PA} + y\vec{PB} + z\vec{PC}$ . Evidently, the next inequality

$$\left(x\vec{PA} + y\vec{PB} + z\vec{PC}\right)^2 \geq 0 \quad (3)$$

holds true.



Because of the properties of the scalar product, this inequality (3) has the form

$$x^2|PA|^2 + y^2|PB|^2 + z^2|PC|^2 + 2xy\overrightarrow{PA} \cdot \overrightarrow{PB} + 2yz\overrightarrow{PB} \cdot \overrightarrow{PC} + 2zx\overrightarrow{PC} \cdot \overrightarrow{PA} \geq 0. \quad (4)$$

By using the law of cosines, we get the equalities

$$\left. \begin{aligned} 2\overrightarrow{PA} \cdot \overrightarrow{PB} &= |PA|^2 + |PB|^2 - c^2, \\ 2\overrightarrow{PB} \cdot \overrightarrow{PC} &= |PB|^2 + |PC|^2 - a^2, \\ 2\overrightarrow{PC} \cdot \overrightarrow{PA} &= |PC|^2 + |PA|^2 - b^2. \end{aligned} \right\} \quad (5)$$

Now from (4) and (5), we obtain:

$$(x^2 + xy + xz)|PA|^2 + (y^2 + yx + yz)|PB|^2 + (z^2 + zx + zy)|PC|^2 - yza^2 - zxb^2 - xyc^2 \geq 0. \quad (6)$$

Finally, if we divide the inequality (6) by  $x + y + z > 0$ , we get the inequality (1).

Evidently, the equality holds if and only if  $x\overrightarrow{PA} + y\overrightarrow{PB} + z\overrightarrow{PC} = \vec{0}$ , i.e.  $x\overrightarrow{PA} + y(\overrightarrow{PA} + \overrightarrow{AB}) + z(\overrightarrow{PA} + \overrightarrow{AC}) = \vec{0}$ , and from here we get (2) after arrangement.

In the sequel we propose several examples of application of the inequality (1).

**Example 1.** For any point  $P$  in the plane of the triangle  $\triangle ABC$  the following inequality holds true:

$$|PA|^2 + |PB|^2 + |PC|^2 \geq \frac{a^2 + b^2 + c^2}{3}. \quad (7)$$

**Solution:** This inequality follows directly from (1) when  $x = y = z = 1$ .

Accounting for (2), the equality holds in (7) if and only if

$$\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AA_1},$$

where  $A_1$  is the midpoint of the side  $BC$ . It means that the point divides the median  $AA_1$  in ratio 2 : 1 computed from the vertex of the triangle, i.e. the point  $M$  is the centroid of the triangle.

**Example 2.** In every triangle  $\triangle ABC$  the following inequality holds true:

$$a^2 + b^2 + c^2 \leq 9R^2. \quad (8)$$

**Solution:** Put  $P \equiv O$  in inequality (1), where the point  $O$  is the circumcenter of the triangle  $\triangle ABC$ . Since  $|OA| = |OB| = |OC| = R$ , now it follows from (7) that

$$3R^2 \geq \frac{a^2 + b^2 + c^2}{3}, \text{ i.e.}$$

$$a^2 + b^2 + c^2 \leq 9R^2, \text{ q.e.d.}$$

The equality holds in (8) if and only if  $a=b=c$ , i.e. for equilateral triangle.

**Example 3.** For anyone point  $P$  in the plane of the triangle  $\triangle ABC$  the following inequality holds true:

$$a|PA|^2 + b|PB|^2 + c|PC|^2 \geq abc. \quad (9)$$

**Solution:** The proof follows directly from (1) when  $x=a, y=b, z=c$ .  
Because of (2), the equality in (9) holds if and only if

$$\overrightarrow{AP} = \frac{b}{a+b+c} \overrightarrow{AB} + \frac{c}{a+b+c} \overrightarrow{AC} = \frac{bc}{a+b+c} \left( \frac{\overrightarrow{AB}}{c} + \frac{\overrightarrow{AC}}{b} \right).$$

It follows now that the vector  $\overrightarrow{AP}$  is collinear with the angular bisector. Analogously, it follows that the vectors  $\overrightarrow{BP}$  and  $\overrightarrow{CP}$  are collinear with the corresponding angular bisectors. Therefore,  $P \equiv I$ , where  $I$  is the incenter.

**Example 4. (Euler's inequality)** In every triangle  $\triangle ABC$  the following inequality holds true:

$$R \geq 2r. \quad (10)$$

**Solution:** Let  $P \equiv O$ , where  $O$  is the circumcenter, i.e.  $|PA|=|OA|=R$ ,  $|PB|=|OB|=R$  and  $|PC|=|OC|=R$ . Now it follows from (9) that

$$R^2(a+b+c) \geq abc$$

$$\Rightarrow R^2 \geq \frac{abc}{a+b+c},$$

and from here using the formulas  $abc=4Rr$  and  $a+b+c=2s$  we obtain:

$$R^2 \geq \frac{4Rrs}{2s}, \text{ i.e.}$$

$$R \geq 2r, \text{ q.e.d.}$$

The equality in (10) holds for  $a=b=c$ , i.e. for the equilateral triangle.

**Example 5.** For any point  $P$  in the plane of the triangle  $\triangle ABC$  the following inequality holds true:

$$\sin 2\alpha |PA|^2 + \sin 2\beta |PB|^2 + \sin 2\gamma |PC|^2 \geq 2F. \quad (11)$$

**Solution:** Put  $x = \sin 2\alpha$ ,  $y = \sin 2\beta$ ,  $z = \sin 2\gamma$  in (1). The right hand side of the inequality (1) takes the form:

$$\frac{a^2 \sin 2\beta \sin 2\gamma + b^2 \sin 2\gamma \sin 2\alpha + c^2 \sin 2\alpha \sin 2\beta}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma},$$

Applying the law of sines formulas  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ ,  $\sin 2\beta = 2 \sin \beta \cos \beta$ ,  $\sin 2\gamma = 2 \sin \gamma \cos \gamma$

$$\frac{16R^2 \sin \alpha \sin \beta \sin \gamma (\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma)}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}$$

and the identities

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma$$

and

$$\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma = \sin \alpha \sin \beta \sin \gamma,$$

the right hand side of the inequality (1) takes the form

$$4R^2 \sin \alpha \sin \beta \sin \gamma.$$

Finally, observe that

$$4R^2 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{2R} = 2F.$$

Consequently, the inequality (11) is true because it follows from the inequality (1).

The equality in (11) holds if and only if  $P \equiv O$ , which is left from for the reader to prove it.

**Example 6.** Let  $P$  be an arbitrary point in the interior of the triangle  $\triangle ABC$ . Prove the inequality

$$\frac{|PA|^2}{c} \left( \frac{1}{a} + \frac{1}{b} \right) + \frac{|PB|^2}{a} \left( \frac{1}{b} + \frac{1}{c} \right) + \frac{|PC|^2}{b} \left( \frac{1}{c} + \frac{1}{a} \right) \geq 2. \quad (12)$$

**Solution:** This inequality is evidently equivalent to the inequality

$$(a+b)|PA|^2 + (b+c)|PB|^2 + (a+c)|PC|^2 \geq 2abc. \quad (13)$$

We will now use the inequality (1).

If  $x = a$ ,  $y = b$ ,  $z = c$  and  $x = b$ ,  $y = c$ ,  $z = a$ , we obtain two inequalities:

$$a|PA|^2 + b|PB|^2 + c|PC|^2 \geq abc$$

and

$$b|PA|^2 + c|PB|^2 + a|PC|^2 \geq abc.$$

Summing the two inequalities, we obtain the following inequality

$$|PA|^2(a+b) + |PB|^2(b+c) + |PC|^2(a+c) \geq 2abc,$$

Which in fact is the inequality (13), thus proving (12).

The equality holds in (12) if and only if  $a=b=c$ , i.e. for equilateral triangle.

## NOTES

1. Murray Klamkin (1921 – 2004) is a Canadian mathematician, born in USA

## REFERENCES

- Arslanagić, Š. (2005). *Matematika za nadarene*. Sarajevo: Bosanska riječ.
- Bottema, O., R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović & P. M. Vasić. (1969). *Geometric Inequalities*. Groningen: (The Netherlands) Wolters-Noordhoff Publishing.
- Grozdev, S. (2007). *For High Achievements in Mathematics: The Bulgaria Experience (Theory and Practice)*. Sofia: ADE (ISBN 978-954-92139-1-1).
- Klamkin, M. S. (1975). Geometric Inequalities via the Polar moment of Inertia. *Math. Mag.*, 48, 44 – 46.
- Sergeeva, T., M. Shabanova & S. Grozdev (2014). *Foundations of Dynamic Geometry*. Moscow: ASOU (in Russian).

✉ **Prof. Šefket Arslanagić, DSc.**

Faculty of Mathematics and Natural Sciences  
University of Sarajevo  
Sarajevo, Bosnia and Herzegovina  
E-mail: asefket@pmf.unsa.ba

✉ **Ms. Daniela Zubović**

Faculty of Mathematics and Natural Sciences  
University of Sarajevo  
Sarajevo, Bosnia and Herzegovina