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THE POWER OF A POINT — A VECTOR PERSPECTIVE

Assoc. Prof. Dr. Boyko Bantchev
Institute of Mathematics and Informatics – Bulgarian Academy of Sciences

Abstract. Vector algebra is a very effective calculation language for doing geometry. It is expressive and succinct, and tends to foster generality and simplicity. In this article we consider from a vector perspective a series of problems concerning circles. After presenting a simple but so far seemingly unnoticed property of the notion of power of a point, we show its application to constructing solutions to the problems.

Keywords: vector algebra; power of a point

1. The power and the intersecting chords and secants

The power $\pi(A)$ of a point A with respect to a circle with centre O and radius R is the number $OA^2 - R^2$. As $\pi(A)$ is positive, vanishes, or is negative depending on A lying outside the circle, on the circumference, or inside the circle, the power expresses the position of a point with respect to a circle.

Of course, if one only needs a test of where a point is with respect to a circle, one can use e.g. $OA - R$ instead of the seemingly more involved $OA^2 - R^2$. What is it that makes the latter expression in fact more useful than the former?

The term *power of a point* was introduced in 1826 by J. Steiner (Steiner 1901), who used it to prove a series of theorems on circles, specifically ones regarding radical centres and radical axes. However, the respective value has been known for a long time. It is known to have emerged in a number of problems related to circles, most notably in the intersecting chords and intersecting secants theorems. Both theorems assert the invariance of the products of the lengths of line segments obtained by intersecting a circle and a line through a point, the product being precisely the power of that point.

These two theorems are indeed a very old and notable part of the edifice of geometry as we know it. They are the subject of several propositions in Book III of Euclid's Elements (Euclid 2002), put in a modern language as follows.

Proposition 35: If the chords AB and CD in a circle cut each other at a point P, then $AP.PB = CP.PD$.

Proposition 36: If a point P is taken outside a circle, and a line from P cuts the circle at points A and B, and another line from P touches the circle at a point T, then $AP.PB = PT^2$.

Proposition 37: If a point P is taken outside a circle, and a line from *P* cuts the circle at points A and B, and another line from P has a point T in common with the circle, and if further $AP.PB = PT^2$, then the second line touches the circle.

As one can observe, Euclid merely establishes the said invariance of the product without mentioning what the product actually is, but the latter is quite apparent.

In modern times, the intersecting chords and intersecting secants theorems are usually proven by making use of inscribed angles and similar triangles. Although similar technique is used to prove both theorems, two different proofs are necessary (Altshiller-Court 2007, Hartshorne 2005).

Euclid himself abstained from using inscribed angles, giving preference to multiple use of Pythagoras' theorem.

For proving the intersecting chords and intersecting secants theorems, Pedoe (Pedoe 1988) resorts to polar coordinates of points and circle equation in Cartesian coordinates. By doing this, he avoids having to present similar but nevertheless different proofs through uniting the 'chord' and 'secant' cases into one theorem. The proof itself, however, is somewhat artificial.

In the following we present an approach to working with circles using vector algebra, and show that the power of a point emerges naturally in this context. The vector-based approach not only allows for unification of the intersecting chords and intersecting secants theorems in a most natural way, but also makes easily apparent another manifestation of the power, so far seemingly unnoticed. The utility of the latter is demonstrated by putting it to use in solving several problems.

2. Circle equations through power

A circle is defined as the set of points in the plane at the same distance R to a given point O, O thus being the centre, and R the radius of the circle. Hence an equation of the circle (O, R) in the variable point P is $OP = R$. As the distance OP is not very useful algebraically, squaring suggests itself: $OP^2 = R^2$, this form of the equation being equivalent to $\pi(P) = 0$. At this point, making use of $OP^2 = \mathbf{OP}^2$, we introduce vectors, so that the equation becomes

$$\mathbf{OP}^2 = R^2$$

where $\mathbf{OP}^2 = \mathbf{OP} \cdot \mathbf{OP}$ is the scalar square of the vector $\mathbf{OP}$.

For the power function itself $\pi(P) = \mathbf{OP}^2 - R^2$ holds, and in case an equation of a disk is needed rather than a circle, we replace $=$ with $\leq$ above.

The vector expression of the equation, in turn, allows making the P's role as the variable point more prominent by substituting $\mathbf{OP} = \mathbf{P} - \mathbf{O}$, thus obtaining

$$\mathbf{P}^2 - 2\,\mathbf{O} \cdot \mathbf{P} + (\mathbf{O}^2 - R^2) = 0.$$

Conversely, any equation of the form

$$\mathbf{P}^2 - 2\,\mathbf{u}\cdot\mathbf{P} + s = 0,$$

where $\mathbf{u}^2 > s$, is an equation of the circle whose centre is the point with position vector $\mathbf{u}$ and whose radius is $\sqrt{\mathbf{u}^2 - s}$.

We see that the 'power' form of the circle or disk equation enables introducing vectors and obtaining yet another, perhaps more manageable form.

Now let A and B be different points and M be the midpoint of AB (Fig. 1). For an arbitrary point P:

$$\mathbf{PA} = \mathbf{PM} + \mathbf{MA}, \quad \mathbf{PB} = \mathbf{PM} + \mathbf{MB} = \mathbf{PM} - \mathbf{MA},$$

hence $\mathbf{PA}\cdot\mathbf{PB} = \mathbf{MP}^2 - \mathbf{MA}^2 = MP^2 - MA^2$.

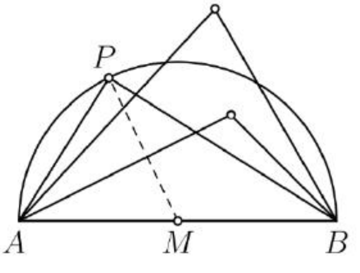

Figure 1

As MA is the radius of the circle with centre M and diameter AB, $MP^2 - MA^2$ is none other than the power $\pi(P)$ of P with respect to this circle, and we obtain the following result.

For any diameter *AB* of a given circle and any point *P* in the plane

$$\mathbf{PA}\cdot\mathbf{PB} = \pi(P) \tag{1}$$

holds: $\mathbf{PA}\cdot\mathbf{PB}$ does not depend on the actual diameter AB and always equals $\pi(P)$.

It also follows that for any diameter AB of a circle

$$\mathbf{PA}\cdot\mathbf{PB} = 0$$

is an equation of the circle (and $\mathbf{PA}\cdot\mathbf{PB} \leq 0$ is an equation of the respective disk).

Yet another conclusion that follows from the above is that, for P different from A and B, as $\mathbf{PA}\cdot\mathbf{PB}$ is positive, zero, or negative according as the angle $\sphericalangle(\mathbf{PA},\mathbf{PB})$ is acute, right, or obtuse, these three cases take place precisely when P

is outside the circle, on the circumference, or inside the circle — a fact that here is a direct consequence of expressing the power of a point through the scalar product of vectors.

3. The power as a scalar product

As we have established the relation (1) for a diameter, the question arises of what other choices for A and B are possible for which (1) is true.

More specifically, let A and B, not necessarily different, both lie on a given circle (O, R), and let P be an arbitrary point. We want to find a necessary and sufficient condition for (1) to hold.

If AA' is a diameter, then from

$$\mathbf{PA} \cdot \mathbf{PB} = \mathbf{PA} \cdot (\mathbf{PA'} + \mathbf{A'B}) = \mathbf{PA} \cdot \mathbf{PA'} + \mathbf{PA} \cdot \mathbf{A'B}$$

and $\mathbf{PA} \cdot \mathbf{PA'} = \pi(P)$ it follows that $\mathbf{PA} \cdot \mathbf{PB} = \pi(P)$ if and only if $\mathbf{PA} \cdot \mathbf{A'B} = 0$. But

$$\mathbf{PA} \cdot \mathbf{A'B} = \mathbf{PA} \cdot (\mathbf{OB} - \mathbf{OA'}) = \mathbf{PA} \cdot (\mathbf{OB} + \mathbf{OA}) = 2\,\mathbf{PA} \cdot \mathbf{OM}$$

where M is the midpoint of AB; if A and B are the same point, then M is also that point. Hence, we find that $\mathbf{PA} \cdot \mathbf{A'B} = 0$ if and only if $\mathbf{PA} \cdot \mathbf{OM} = 0$. Furthermore, since $\mathbf{PA} \cdot \mathbf{OM} = 0$ (because either $M \equiv A$, or $M \equiv O$, or $MA \perp OM$) and $\mathbf{PA} = \mathbf{PM} + \mathbf{MA}$, we also find $\mathbf{PA} \cdot \mathbf{OM} = \mathbf{PM} \cdot \mathbf{OM}$. Thus $\mathbf{PA} \cdot \mathbf{PB} = \pi(P)$ if and only if $\mathbf{PM} \cdot \mathbf{OM} = 0$.

The latter equality holds if and only if either $M \equiv O$ — which is when AB is a diameter — or P is on the line through M that is perpendicular to OM (and possibly $P \equiv M$). If AB is not a diameter and $A \not\equiv B$, then $\mathbf{PM} \cdot \mathbf{OM} = 0$ holds precisely when the secant AB contains P. And if $A \equiv B$ and thus also $M \equiv A$, then, since M is on the circle, $\mathbf{PM} \cdot \mathbf{OM} = 0$ means that P lies on the tangent at M.

We conclude that (1) holds in precisely two cases: when AB is a diameter, and when A and B are the points (not necessarily different) that a line through P has in common with the circle (O, R). In particular, we have found that $PA^2 = \pi(P)$ if and only if A is the contact point of the tangent throught P (and therefore $A \equiv B$).

As it follows that for a secant AB through P the product $\mathbf{PA} \cdot \mathbf{PB}$ depends only on P and not on A and B, we have established both the intersecting chords and the intersecting secants theorems, along with the liminal case when a line is tangent rather than a secant. All the three cases are established at once, with no need for separate proofs. Fig. 2 shows examples for P outside and inside a circle.

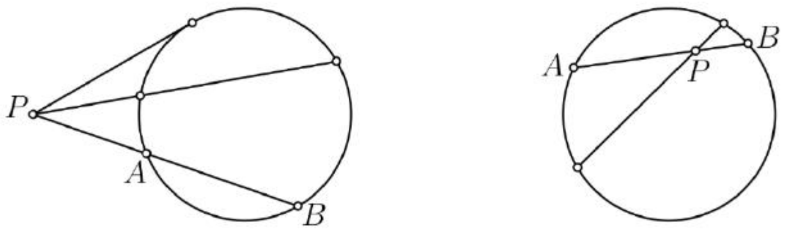

Figure 2

Note that the power invariance seems to manifest itself more immediately in the diameter case than the chord/secant one, and that we have used the former in deriving the latter. And the fact that we use a scalar product rather than just the length product $PA.PB$, as is commonly done, makes it possible to not differentiate between P lying inside or outside the circle, as well as treat the diameter and the chord/secant cases jointly.

4. Finding the other end of a chord

The diameter case (1) appears to help in solving various problems. One of them is finding the other point of intersection of a line and a circle, given one such point.

Let K be a point of intersection of a line through the point $A \not\equiv K$ with some circle. The line AK may also intersect the circle at another point L, or it may be that AK touches the circle at K, in which case we assume $L \equiv K$.

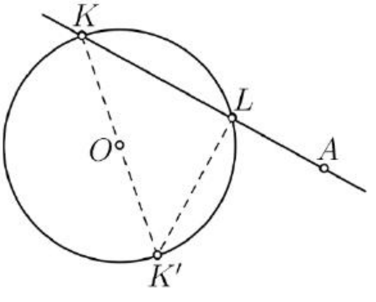

Figure 3

Let K' be the other end of the diameter through K (Fig. 3). If $L \not\equiv K'$, then $K'L \perp AK$, because KK' is a diameter and L lies on the circle. This is true even if $L \equiv K$. Therefore L is the perpendicular projection of K' on AK, and as the latter remains true when $L \equiv K'$, L is defined by

$$\mathbf{L} = \mathbf{A} + \frac{\mathbf{AK} \cdot \mathbf{AK}'}{AK^2} \mathbf{AK}$$

As $\mathbf{AK} \cdot \mathbf{AK}'$ is the power of A with respect to the circle, the product is independent of K and equals $OA^2 - R^2$, where O and R are the centre and radius of the circle. Hence L can also be defined by

$$\mathbf{L} = \mathbf{A} + \frac{OA^2 - R^2}{AK^2} \mathbf{AK}.$$

Thus, knowing any of the points in common of a line through A and a circle, the other point can be found by any of the above equalities: O being given, we also know $\mathbf{AK}' = \mathbf{AO} - \mathbf{OK}$ and $R^2 = OK^2$. If the line is tangent to the circle and K is the only point in common, the equalities produce an L that is the same as K.

For another example of the kind, let the point K be on the line through the side BC of ΔABC, and the line AK has another point L in common with the circumcircle of ΔABC (Fig. 4). L is, in the general case, different from A. How can we find L?

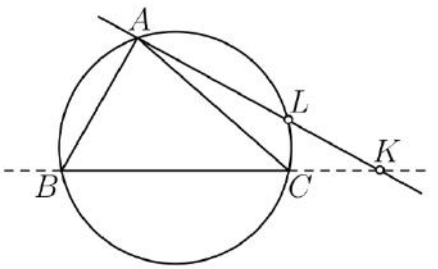

Figure 4

Let at first indeed $L \not\equiv A$. As long as K belongs to each of the lines AL and BC and the points A, B, C, and L are on the circumcircle of ΔABC, the power of K with respect to that circle can be expressed through A and L, as well as through B and C, which leads to $\mathbf{KL} \cdot \mathbf{KA} = \mathbf{KB} \cdot \mathbf{KC}$.

The same equality holds when $L \equiv A$.

Since $\mathbf{KL} \| \mathbf{KA}$, there must be a number s such that $\mathbf{KL} = s\,\mathbf{KA}$. Substituting this in the first equality and thus finding s, we arrive at the following expression for L:

$$\mathbf{L} = \mathbf{K} + \frac{\mathbf{BK} \cdot \mathbf{KC}}{AK^2} \mathbf{AK}.$$

As a point on the line BC, the point K may be known through the coefficient λ in $\mathbf{K} = \mathbf{B} + \lambda\,\mathbf{BC}$. In this case the above equality takes the form

$$\mathbf{L} = \mathbf{K} + \lambda(1-\lambda)\left(\frac{BC}{AK}\right)^2 \mathbf{AK}.$$

5. A formula of Euler and its likes

Expressing the power of a point through a diameter as in (1) can also be put to service in another way.

Let P be any point and K be a point on the circle (O, R). Of the power of P with respect to the circle we know $OP^2 - R^2 = \mathbf{PK} \cdot \mathbf{PK}'$, where K' is the other end of the diameter through K. This equality can be used to find OP indirectly, through K. In order to get rid of K', we transform the expression on the right-hand side as follows:

$$\mathbf{PK} \cdot \mathbf{PK}' = \mathbf{PK} \cdot (\mathbf{PK} + \mathbf{KK}') = \mathbf{KP} \cdot (\mathbf{KP} + 2\,\mathbf{OK})$$

thus obtaining for OP

$$OP^2 = R^2 + \mathbf{KP} \cdot (\mathbf{KP} + 2\,\mathbf{OK}) = R^2 + KP^2 + \mathbf{KP} \cdot (2\,\mathbf{OK}).$$

The above equality is useful for finding OP when it is difficult to do it directly but for some K we can find $\mathbf{OK}$ and $\mathbf{KP}$. For K we have a choice — it can be any point on the circle.

Typically, the circle is the circumcircle of a triangle and K is any of its vertices. Let the triangle be ΔABC and A be taken for the role of K.

In the following, we use the conventional notation for the sides of the triangle: $a = BC$, $b = CA$, and $c = AB$. We also designate the sides as vectors: $\mathbf{a} = \mathbf{BC}$, $\mathbf{b} = \mathbf{CA}$, and $\mathbf{c} = \mathbf{AB}$. Let also S be the doubled oriented area of ΔABC.

The centre O of the circumcircle can be found as the point of intersection of the perpendicular bisectors of any two of the sides of ΔABC, e.g. CA and AB:

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} - \frac{1}{2}\frac{\mathbf{b} \cdot \mathbf{c}}{S}\mathbf{a}^{\perp}$$

where $\mathbf{a}^{\perp}$ is the vector the same length as $\mathbf{a}$ and rotated to a right angle in the positive direction with respect to $\mathbf{a}$.

With this and $K \equiv A$, $\mathbf{OK}$ becomes

$$\frac{\mathbf{b} - \mathbf{c}}{2} + \frac{1}{2}\frac{\mathbf{b} \cdot \mathbf{c}}{S}\mathbf{a}^{\perp}$$

and the above expression for $\mathbf{OP}$ takes the form

$$OP^2 = R^2 + AP^2 + \mathbf{AP} \cdot \left(\frac{\mathbf{b} \cdot \mathbf{c}}{S}\, \mathbf{a}^{\perp} + \mathbf{b} - \mathbf{c}\right). \tag{2}$$

The equality (2) helps expressing the distance of a number of triangle centres to O.

For example, let P be the centroid G of ΔABC. Applying $\mathbf{AG} = (\mathbf{c} - \mathbf{b})/3$, $(\mathbf{c} - \mathbf{b}) \cdot \mathbf{a}^{\perp} = -2\,S$, and (2) yields

$$OG^2 = R^2 - \frac{2}{9}\,(b^2 + c^2 + \mathbf{b} \cdot \mathbf{c}).$$

In view of $\mathbf{a} = -(\mathbf{b} + \mathbf{c})$ the expression in the parentheses can be slightly tightened to $a^2 - \mathbf{b} \cdot \mathbf{c}$. If, on the other hand, a more symmetric with respect to a, b, c expression is sought, we can arrive at the following:

$$OG^2 = R^2 - \frac{a^2 + b^2 + c^2}{9}.$$

If H is the orthocentre, taking into account $\mathbf{OH} = 3\,\mathbf{OG}$ we also obtain

$$OH^2 = 9\,OG^2 = 9\,R^2 - (a^2 + b^2 + c^2).$$

Now let P be the incentre I of ΔABC and again make use of (2). Being the point of intersection of the angle bisectors at B and C of ΔABC, I is defined by

$$\mathbf{I} = \mathbf{A} + \frac{b\,\mathbf{c} - c\,\mathbf{b}}{p}$$

where $p = a + b + c$, and (2) becomes, upon doing the multiplication,

$$OI^2 - R^2 = \frac{(b\,\mathbf{c} - c\,\mathbf{b})^2}{p^2} - \frac{b\,c^2 + c\,b^2}{p} = \frac{b\,c}{p^2}\left(2\,(b\,c - \mathbf{b} \cdot \mathbf{c}) - p\,(b + c)\right).$$

Here we substitute $p = a + b + c$ to obtain

$$-\frac{b\,c}{p^2}\left((\mathbf{b} + \mathbf{c})^2 + a\,(b + c)\right),$$

from where, in turn, by substituting $\mathbf{b} + \mathbf{c} = -\mathbf{a}$, we arrive at

$$-\frac{a\,b\,c}{p^2}\,(a + b + c) = -\frac{a\,b\,c}{p}.$$

Due to $pr = |S|$ where r is the inradius, and also $abc = 2R|S|$, the above expression turns to be equal to $-2Rr$, thus

$$OI^2 = R^2 - 2Rr. \tag{3}$$

Similar calculation for any excentre, e.g. I_a — the one against A and defined by

$$\mathbf{I_a} = \mathbf{A} + \frac{b\,\mathbf{c} - c\,\mathbf{b}}{-a + b + c} = \mathbf{A} + \frac{b\,\mathbf{c} - c\,\mathbf{b}}{p - 2\,a}$$

leads to

$$OI_a^2 = R^2 + 2Rr_a,$$

where r_a is the respective inradius.

The formula (3) is widely known as 'Euler's triangle formula'. In (Altshiller-Court 2007) and elsewhere it is proven by combining similar triangles, the chord intersection theorem, and a somewhat involved equality of line segments.

The vector-based approach that we demonstrated here is more straightforward. Moreover, our derivation is but one of the many useful applications of (2), and (2) in turn is one of the uses of the diameter-related expression (1) of the power of a point concept. Being directly related to the use of vectors, (1) is an example of the fruitfulness of applying vector algebra to solving problems in geometry. And again it is vector algebra that makes possible the unified treatment of the chords/secants theorems, as well as various generalizations, a minuscule part of which is shown in this article.

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✉ **Dr. Boyko Banchev, Assoc. Prof.**
ORCID iD: 0000-0002-0284-312X
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev str., bl. 8
1113 Sofia, Bulgaria
E-mail: boykobb@gmail.com