ONE GENERALIZATION OF THE GEOMETRIC PROBLEM FROM 19TH JUNIOR BALKAN MATHEMATICAL OLYMPIAD

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Abstract. We present one possible generalization, inspired by the usage of the Dynamic Geometry Software GeoGebra, of the geometric problem from the 19th Junior Balkan Mathematical Olympiad. We have presented the process of the generalization in front of students about 16 – 17 years of age from the Mathematical School "Academic Kiril Popov" – Plovdiv. We have use the technique of tree diagrams to ease students' understanding of the solution and to help them in the steps of the generalization.

Keywords: Math Olympiad; Dynamic geometry software; Experiment; Tree diagram

1. Introduction

The discussion on a geometry problem, its proof and its possible generalizations involves oral presentation and pointing at different parts of the figure of the problem. Students have to watch and listen simultaneously. They have to refer to many elements and incorporate them into their memory, as pointed in (Sweller, 1988). The Dynamic Geometry Softwares (DGS) are a valuable tool for its users to facilitate the above mentioned problems. DGS can significantly optimize the teaching process (Karaibryamov & al., 2012), (Karaibryamov & al., 2013), (Tsareva & Zlatanov, 2016) and increase its creative elements (Taneva, 2015), (Trifonova & al., 2013), (Tsareva & Todorova, 2013), (Zlatanov, 2014), (Zlatanov, 2017).

A geometry problem is specified with a verbal description, often accompanied by a figure, which was discussed in (Wong & al.,2011). As (Mayer & Sims, 1994) pointed out, students can build more referential connections when verbal and visual materials are presented contiguously than when they are presented separately. It is also noted in (Clements & Battista, 1992) that for a student to successfully prove a problem, he/she must build semantic links between the concepts of geometry and the features of a figure. Through bi-directional connections, students can clearly demonstrate the interrelation between the geometric components in a verbal description and the objects in an accompanied figure (Schnotz, 2002) integrative model of text (descriptive representation) and picture (depictive representation) comprehension emphasizes that good graphic design is crucial for individuals with low prior knowledge who need pictorial support in constructing mental models. The use of DGS in math education gives positive results both in students in math focused high-schools or in ordinary schools and in teaching of bilinguals (Grozdev & al.,2014). DGS strongly helps research work simply because it allows the experimental discovery of new relations between the investigated objects, which leads to their formal proof (Nenkov, 2010), (Grozdev & Nenkov, 2000), (Grozdev & Nenkov, 2015), (Zlatanov, 2018). This style of teaching often leads to interesting generalizations and discoveries of new geometric objects, connected with the studied geometric configuration (Grozdev & Nenkov, 2012), (Grozdev & Nenkov, 2014), (Grozdev & Nenkov, 2017). In the cases when the assumptions do not confirm experimentally by DGS, a lot of time and efforts are saved.

We would like to mention a very powerful computer software "Discover" for discovering of new relationships in geometry (Grozdev, Okumura & Dekov, 2018a), (Grozdev, Okumura & Dekov, 2018a).

2. Generalization of geometric problems

Following (Wong & al., 2011) let us say a few words about the techniques of proving of geometry problems. The steps needed to make a formal proof of a geometry problem is presented in Figure 1 (Wong & al., 2011). A student can interact with a dynamic figure, which provides a clear picture of abstract mathematical ideas through concrete object dragging. By manipulating a dynamic figure and observing how it changes, students may be able to avoid over-generalization of theorems from paradigmatic images. DGS allows students to either falsify propositions or enhance the degree to which propositions are believable.

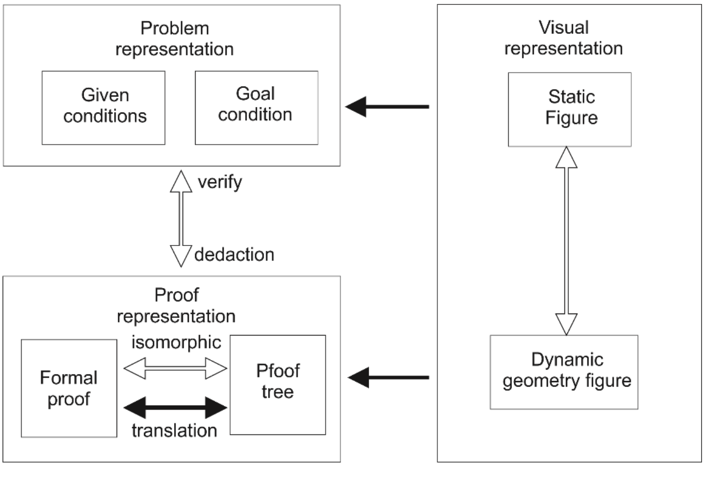

Figure 1

Once the problem is solved, students can start to use the dynamic of the sketch to search for generalizations. We have tried in Figure 2 to expand Figure 1 in the case, when students will be asked to search for a generalization of an already solved problem.

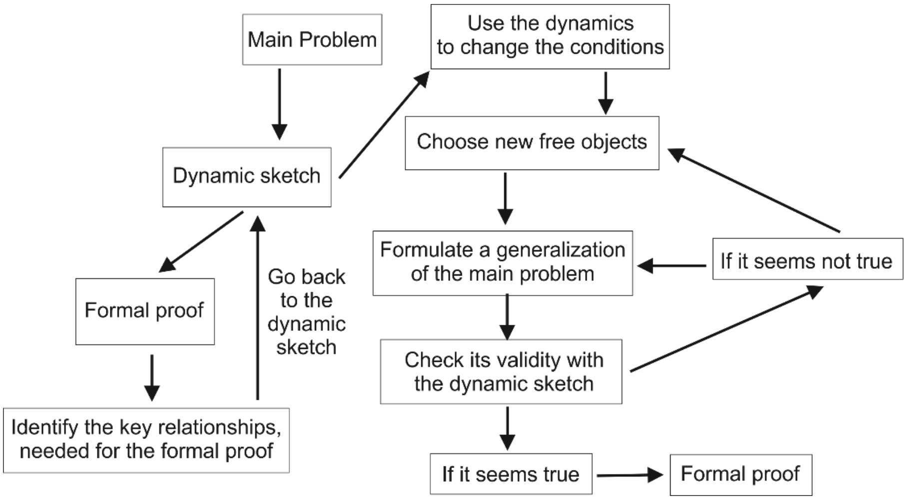

Figure 2

Once a formal proof is made, it is important to identify the key relationships that were needed in the proof. The choice of the free object in any DGS sketch is important (Karaibryamov & al., 2013). Sometimes it happens that a new sketch is needed to be done in order to choose new free objects. The new sketch will give the student greater dynamic options, which will increase the possibility to find generalizations.

We will illustrated this approach by a generalization of a nontrivial problem.

We introduced a group of 9th and 10th graders of High School of Mathematics "Akademik Kiril Popov", Plovdiv, Bulgaria with the specific powerful tools of GeoGebra and we presented them the following geometric task from 19th Junior Balkan Mathematical Olympiad June 24-29, 2015, Belgrade, Serbia:

Problem 1. Let ΔABC be an acute triangle. The lines l_1 and l_2 are perpendicular to AB at the points A and B, respectively. Now let h and g be the perpendicular lines from the midpoint M of the side AB to the lines AC and BC. Let h and g intersect the lines l_1 and l_2 at the points E and F, i.e. $h \cap l_1 = E$ and $g \cap l_2 = F$. Let D be the intersection point of the lines EF and MC. Prove that $\angle ADB = \angle EMF$ (Figure 3).

To ease the reader we will mark the main steps of the solution to this nice problem. From the condition it follows that $MHCG$ is a cyclic quadrilateral. Thus we have

(1) $$\angle HMC = \angle HGC$$

From the similar rectangular triangles ΔMHA and ΔMAE, ΔMBG and ΔMFB follow the equalities $MA:ME = MH:MA$ and $MB:MF = MG:MB$, which can also be written as $MA^2 = MH.ME$ and $MB^2 = MG.MF$. If we take into account that $MA = MB$ we get $EHGF$ is a cyclic quadrilateral and then

(2) $$\angle FEH = \angle FEM = 180^0 - \angle HGF = \angle HGM$$

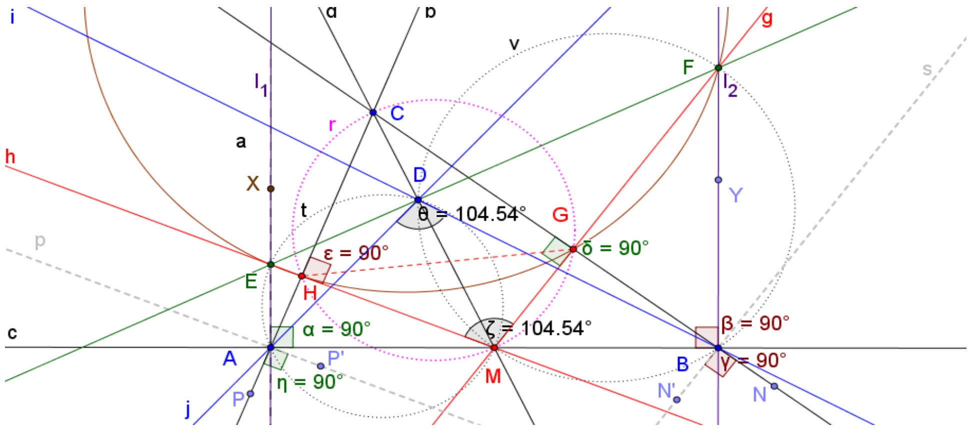

Figure 3

From (1) and (2) we have that $\angle DEM + \angle EMD = \angle HGM + \angle HGC = 90^0$ or $CM \perp EF$ and therefore $MAED$ and $MBFD$ are cyclic quadrilaterals. Thus $\angle DEM = \angle DAM$ and $\angle DFM = \angle DBM$. Then ΔEMF and ΔADB are similar. Therefore and the third pair of their angles are equal as well, i.e. $\angle ADB = \angle EMF$.

An interesting idea is presented in (Wong & al.,2011), where a tree diagram of the formal proof is presented. The authors refer to this tree diagram as a proof tree. It oferers an outline of a complete geometry proof. By using a proof tree students could better understand the formal proof. This helps students to develop a better understanding of geometry proofs.

Following (Wong & al.,2011) we can draw the proof tree of Problem 1. We have marked the key steps in the proof in gray. We are searching for a generalization that will not change the conclusions that quadrilaterals *MHCG,EHCG, MAED* and *BFDM* are cyclic.

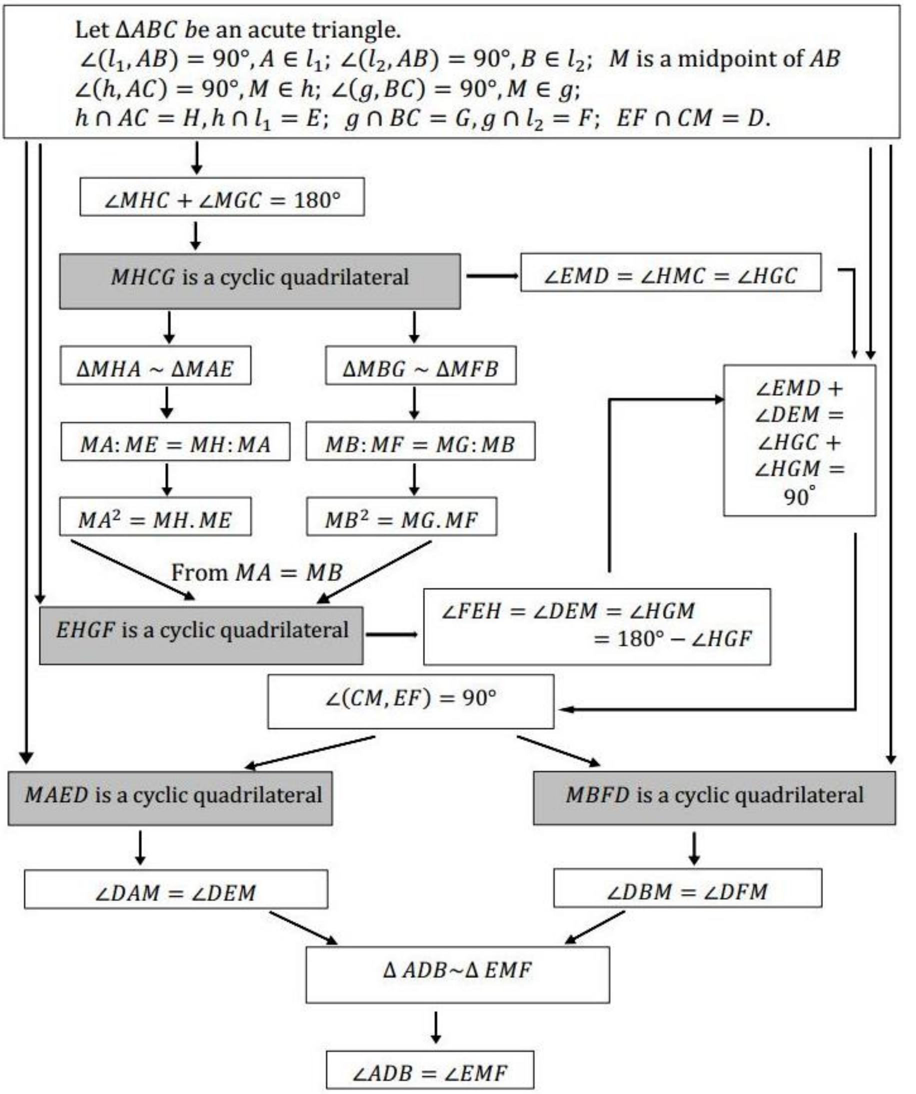

Figure 4

Naturally there occurs the question whether the 90^0 angle formed by l_1 and l_2 intersecting *AB* and the angles formed by h and g intersecting *AC* and *BC* is essential for the statement $\angle ADB = \angle EMF$ or it may be arbitrary.

We have formulated our hypothesis into the next problem:

Problem 2. Let ΔABC be an acute triangle and let the parallel lines l1 and l2 be going through the points *A* and *B* respectively, and be forming an angle α with the line *AB*. Also let *M* be the midpoint of the side *AB*. From point *M* we build the lines *h* and *g*, forming an angle *a* with *AC* and *CB*, respectively. Let us mark the intersecting points of the line *h* with the lines *AC* and l_1, with *H* and *E* and the intersecting points of the line *g* with the lines *BC* and l_2 with *G* and *F*, respectively. If *D* is the intersecting point of *EF* and *MC*, prove that $\angle ADB = \angle EMF$.

Using GeoGebra environment, we created a dynamic drawing (a snapshot of which is Figure 5) reflecting the conditions of Problem 2.

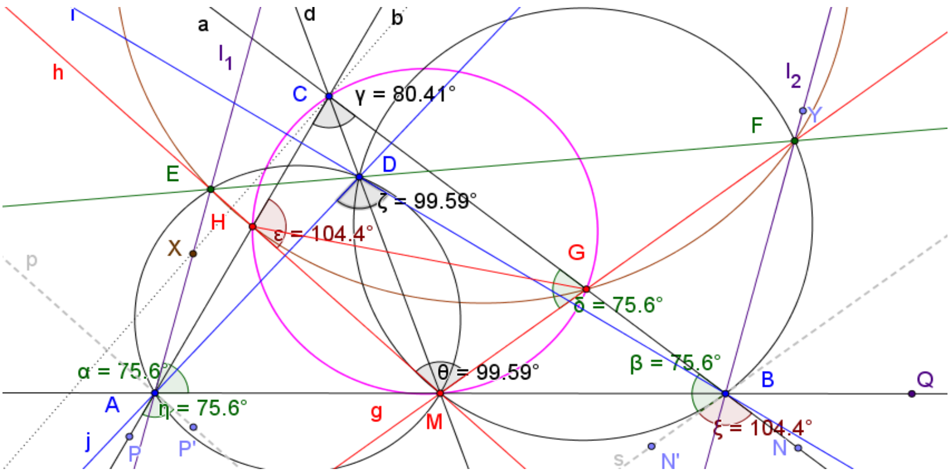

Figure 5

Now let *X* be an arbitrary point in the plain of ΔABC and then l_1 will be the line *AX*, i.e. $l_1 = AX$. Thus

(3) $$\angle XAB = \angle(AX, AB) = \angle(l_1,AB) = a.$$

The construction of the lines *h* and *g* is preceded by the construction of the lines $p = AP'$ and $s = BN'$ with the help of the tool “Angle with given size” such that

(4) $$\angle PAP' a = \angle NBN' = 180^0 - a,$$

to ensure that the lines $h//AP'$ and $g//BN'$ to cross the lines *AC* and *CB* in such a way that the quadrangle *MHCG* to be a cyclic quadrilateral ($\angle MHC = 180^0 - a$, $\angle MGC = a$)

The students experimented by moving the free point X to change the angle α or by moving the peaks of ΔABC to change it, but each time the hypothesis was confirmed. Then we proceeded to the proof of the new statement.

Solution. The experiments outlined the different locations of the points *H, M, E* and G *M,F*. We examined in detail the case when the point *H* is between the points *M* and *E*, i.e.*H/ME* and *G/MF* (Figure 6).

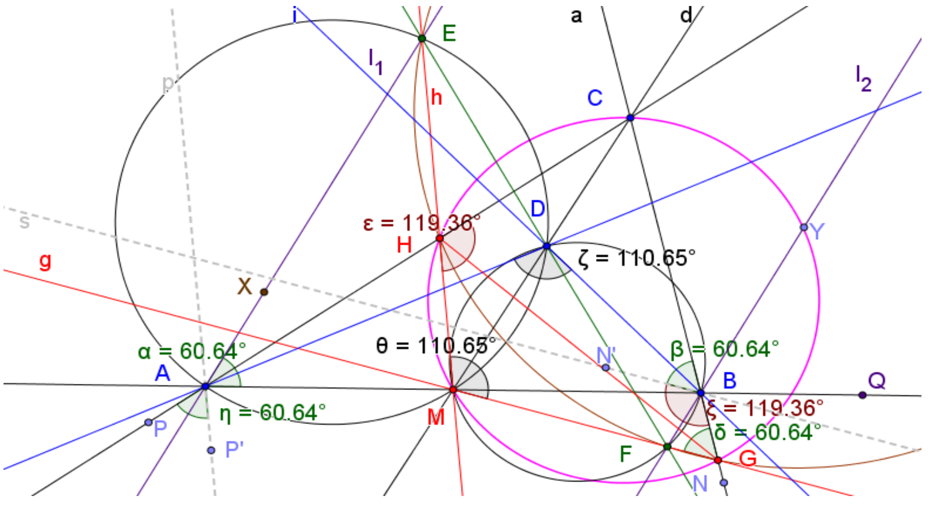

Figure 6

Now the line *h,* passing through the point *M* and parallel to the line *p*, provides the conditions

(5) $$\angle MHA = PAP' = a$$

and

(6) $$\angle MHC = 180^0 - \angle MHA = 180^0 - a,$$

Because $\angle AHM$ and $\angle PAP$ are adjacent angles obtained by crossing the parallel lines *h* and *p* with the line *AB*.

From (3) and (5) it follows that:

(7) $$\angle MAE = \angle MHA = a.$$

From (7) and the presence of a common angle it follows that ΔMHA and ΔMAE are similar. Therefore :*ME* = *MH:MA*, which is equivalent to the equality:

(8) $$MA^2 = MH.\ ME,$$

Now from the properties of the external adjacent angels $\angle MGC$ and $\angle NBN'$ formed by the parallel lines *s* and *g* crossing *BC* we can write

(9) $$\angle MGC = 180^0 - \angle MGB = 180^0 - \angle NBN' = 180^0 - (180^0 - a) = a.$$

From (6) and (9) we have $\angle MHC + \angle MGC = 180^0$, i.e. *MHCG* is a cyclic quadrilateral. Therefore

(10) $\angle HMC = \angle HGC$.

If we look at ΔMBF and ΔMGBwe can see that they are similar because they have one common angle and $\angle$MBF = $\angle$MGB = $180^0 - a$. Therefore, $MB:MG = MF:MB$ which is equivalent to:

(11) $MB^2 = MG.MF$

From (8), (11) and $MA = MB$ it follows $MH.ME = MG.MF$. This sufficient condition allows us to conclude that $EHGF$ is a cyclic quadrilateral. Therefore, $\angle FEH + HGF = 180^0$. At the same time, due to the property of the neighboring angles we have. $\angle HGM + \angle HGF = 180^0$.

Therefore,

(12) $\angle FEH = \angle HGM = 180^0 - \angle$HGF.

From (10), (12) and (9) it follows:

(13) $\angle HMC + \angle FEH = \angle HGC + \angle HGM = \angle MGC = a$.

Then $\angle EMD + \angle DEM = \angle HMC + \angle FEH = a$ and

(14) $\angle MDE = 180^0 - a$.

According to (3) and (14) $MAED$ is a cyclic quadrilateral. Therefore,

(15) $\angle DEM = \angle DAM$.

From (14) and the property of the neighbouring angles we have $\angle MDF = a$. Noting the condition $\angle MBF = 180^0 - a$, we conclude that $MBFD$ is a cyclic quadrilateral. Therefore,

(16) $\angle DFM = \angle DBM$

From (15) and (16) we have that ΔEMF and ΔADB are similar. Then and the third pair of angles are equal as well, i.e. $\angle ADB = \angle EMF$.

After the discussion on the proof of the generalized problem, students were able to write its proof tree:

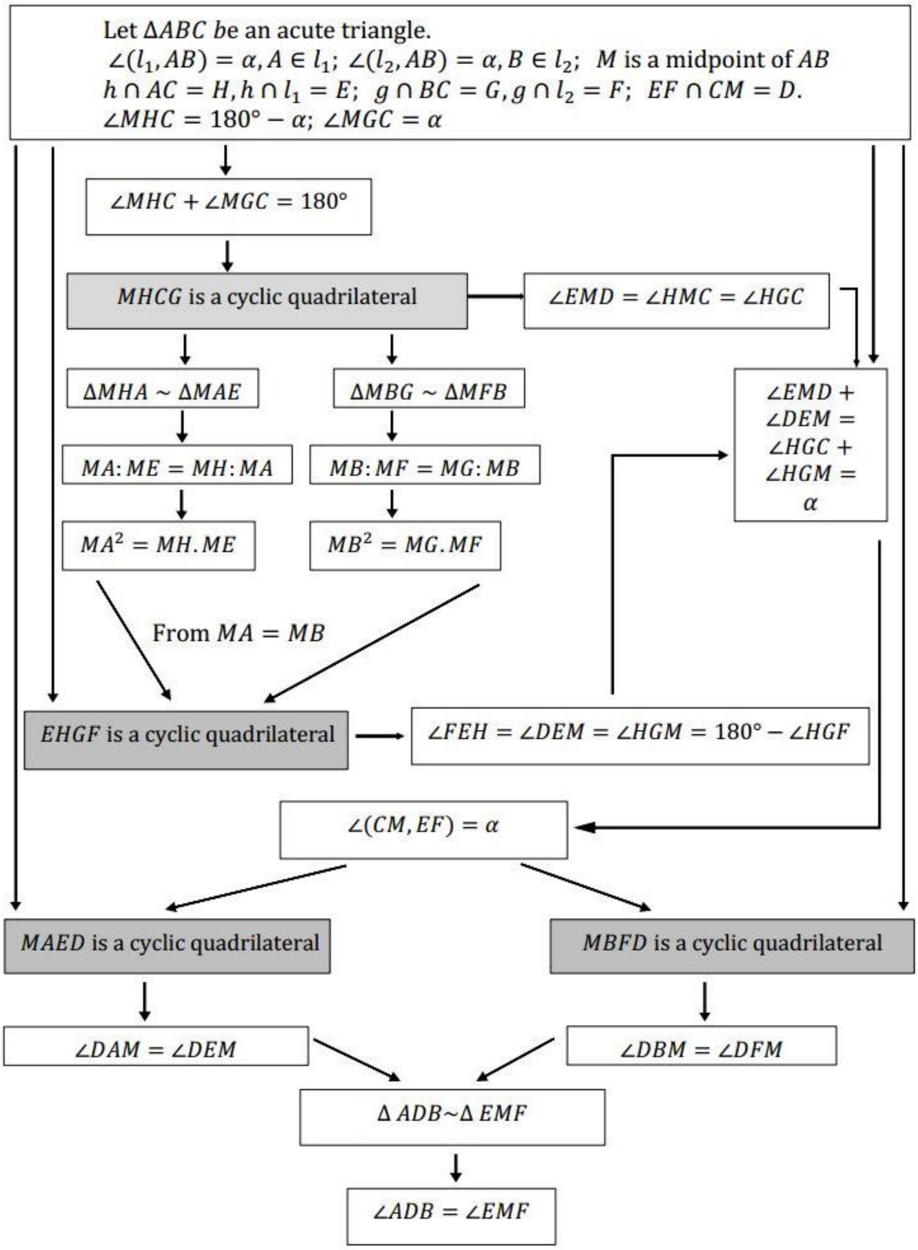
Let ΔABC be an acute triangle.
∠(l1, AB) = α, A ∈ l1; ∠(l2, AB) = α, B ∈ l2; M is a midpoint of AB
h ∩ AC = H, h ∩ l1 = E; g ∩ BC = G, g ∩ l2 = F; EF ∩ CM = D.
∠MHC = 180° − α; ∠MGC = α
∠MHC + ∠MGC = 180°
MHCG is a cyclic quadrilateral
∠EMD = ∠HMC = ∠HGC
ΔMHA ~ ΔMAE
ΔMBG ~ ΔMFB
∠EMD + ∠DEM = ∠HGC + ∠HGM = α
MA: ME = MH: MA
MB: MF = MG: MB
MA2 = MH.ME
MB2 = MG.MF
From MA = MB
EHGF is a cyclic quadrilateral
∠FEH = ∠DEM = ∠HGM = 180° − ∠HGF
∠(CM, EF) = α
MAED is a cyclic quadrilateral
MBFD is a cyclic quadrilateral
∠DAM = ∠DEM
∠DBM = ∠DFM
Δ ADB~Δ EMF
∠ADB = ∠EMF

Figure 7

We also looked at the case where *H/ME* and *F/MG* (Figure 3). Once the students have understand the proof, they were able to write the formal proof alone. Since the proof is analogical, we do not give it to the reader.

In the process of the proof we demonstrated the ability of the "Object properties" window to preserve all constructions and after changing some object parameters. We also familiarized students with other GeoGebra tools, such as "Polygon". They shuddered the cyclic quadrilaterals or the similar triangles to orient more easily in the overwhelmed graph. By setting the angle $a = 90^0$, the students acquired Figure 1, illustrating the problem from the 19th Junior Balkan Mathematical Olympiad June 24-29, 2015, Belgrade, Serbia. There only exists the possibility *H/ME* and *G/MF* in the original problem. We have set a new assignment: Explore whether the condition that the triangle is acute may be dropped.

The following interesting fact was also noticed: With fixed ΔABC, the point D proved to be stationary and not affected by the change of angle α. We discussed this fact and, by clarifying it, we reached the following statements following from the main result:

Corollary 1. The point D is the common point of the median CM of ΔABC with the geometrical place of points, from which the side AB is seen under the angle $180^0 - \angle ACE$.

We believe that the current work, inspected by (Wong & al., 2011), will support the process of building creative thinking in geometry classes.

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ОБОБЩЕНИЕ НА ГЕОМЕТРИЧНАТА ЗАДАЧА ОТ 19-АТА МЛАДЕЖКА БАЛКАНСКА ОЛИМПИАДА ПО МАТЕМАТИКА

Резюме. Представяме едно възможно обобщение, инспирирано чрез използването на динамичния геометричен софтуер GeoGebra, на геометричната задача от 19-ата младежка балканска олимпиада по математика. Представихме процеса на обобщението пред ученици на възраст между 16 и 17 години от Математическата гимназия „Академик Кирил Попов“ – Пловдив. Използвахме техниката на дървовидната диаграма, за да улесним разбирането на решението от страна на учениците и да им помогнем в стъпките на обобщаването.

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