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VOLUME OF A TRUNCATED CONE VIA GEOMETRIC SIMILARITY

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Abstract. The following study addresses the issue of calculating with precision the volume of a truncated right circular cone while provided with limited information on the dimensions of the object itself (being given the relation between the radii, the vertical heights, or the slanted heights). The results include the proof of two theorem generalizations for the calculation of said volume with either of the three given elements via the principles of geometric similarity. It is shown that, due to the similarity between the full cone and the smaller removed cone, the volume of the truncated cone can be expressed using the difference of cubes of the corresponding linear dimensions. Revisiting the classical volume formula through the principles of geometric similarity, this work provides six alternative expressions that have both theoretical value and direct applications, especially in the field of education.

Keywords: Right circular truncated cone; Volume formulas; Geometric similarity; Multi-dimensional parameterization; Educational, applied, and theoretical value

1. Introduction

Calculating the volume of a truncated cone — also known as a conical frustum — is a classical geometric problem of both theoretical and practical significance. Given a right circular cone from which a smaller, similar cone is removed via a cut parallel to the base, the goal is to determine the volume of the remaining solid. Traditionally, this volume is found using a standard formula involving the radii of the two circular bases and the vertical height of the frustum, which is present in secondary school and university geometry textbooks (Milkoeva, 2025; Galabova & Siderova, 2025; Stoyanov, 2020). However, this approach can obscure the geometric reasoning behind the formula and its relation to similarity, and is less applicable in instances in which the necessary elements are not explicitly given.

Historically, formulas for the volume of conical frustums date back to the ancient Greeks, with contributions from Archimedes and later from Renaissance mathematicians. These classical methods often relied on geometric dissection or the method of exhaustion, and are summarized today by the standard expression $V = \frac{\pi h}{3}\left(R^2 + Rr + r^2\right)$. In modern treatments, this formula is typically derived using calculus or by subtracting the volume of the smaller cone from that of the full cone.

In this paper, the problem is revisited through the principles of geometric similarity (Petrov & Petrova, 2018) and the proof of two general formulas with six alternative corollary expressions for the volume of a truncated cone, using the difference of cubes of the radii, of the vertical heights, or of the slant heights. Each of these forms naturally results from observing that the removed small cone is similar to the full cone and that the volume of similar solids scales with the cube of the ratio of corresponding lengths, as first established in Euclid's Elements Book XII (Euclid, 1908).

The problem is of interest not only because of its elegant mathematical structure, but also because the alternative formulas highlight the deep connection between similarity and volume in Euclidean space. Moreover, they provide insights into how geometric reasoning can lead to compact and general expressions, which may be of pedagogical value in both school and university-level mathematics.

This paper is structured as follows: The *Main Body* states the definitions and supports visual intuition through a graph, proves each of the two theorem generalizations in detail, provides six alternative expressions, and offers remarks on the coefficient arising from the difference of cubed dimensions. The following *Examples* part gives instances of use in mathematical problems. Afterwards, there is a short *Applications* section that briefly discusses practical implications of the results. Finally, the *Conclusion* summarizes the findings and their broader relevance.

2. Main Body

2.1. Definitions and Geometric Setup

We consider a right circular cone from which a smaller cone is removed by a plane parallel to the base, resulting in a truncated cone (frustum). Since the cutting plane is parallel to the base, the removed cone is similar to the original one (Figure 1).

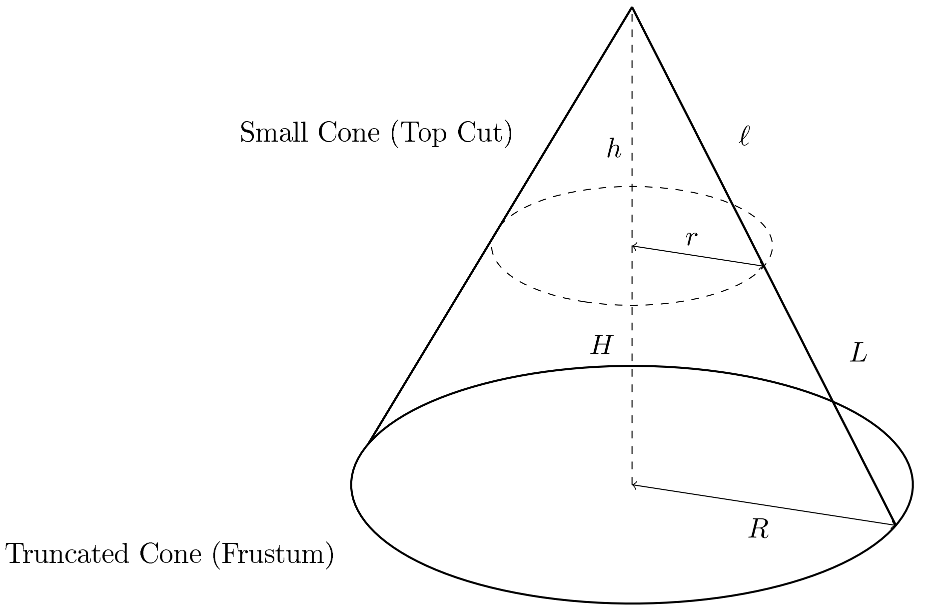

Figure 1. 3D visualization of the figure

Let:

- R be the base radius of the full cone,
- H be the vertical height of the full cone,
- r be the radius of the smaller cone removed from the top,
- h be the height of the smaller cone,
- L be the slant height of the full cone,
- ℓ be the slant height of the smaller cone.

2.2. Theorem Generalization

Theorem A (Volume in terms of the full cone)

Let $A \in \{R, H, L\}$ and $a \in \{r, h, \ell\}$ be corresponding linear dimensions of a right circular cone and a smaller cone removed by a plane parallel to the base. Then the volume of the truncated cone is

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{A^3 - a^3}{A^3}.$$

Proof. Let the full cone have a linear dimension A, and let its volume be denoted by V_{full}. Let a smaller cone, similar to the full one, be removed from the top, having the corresponding linear dimension $a < A$.

Since the cutting plane is parallel to the base, the two cones are similar. Hence, the ratio of any corresponding linear dimensions is constant, and we have

$$\frac{a}{A} = \frac{\text{any linear dimension of the small cone}}{\text{the corresponding dimension of the full cone}}.$$

By similarity, the ratio of the volumes of the two cones is equal to the cube of the ratio of their corresponding linear dimensions. Therefore,

$$\frac{V_{\text{small}}}{V_{\text{full}}} = \left(\frac{a}{A}\right)^3,$$

which gives

$$V_{\text{small}} = V_{\text{full}} \cdot \frac{a^3}{A^3}.$$

The volume of the truncated cone is obtained by subtracting the volume of the smaller cone from the volume of the full cone:

$$V_{\text{frustum}} = V_{\text{full}} - V_{\text{small}} = V_{\text{full}} - V_{\text{full}} \cdot \frac{a^3}{A^3}.$$

Factoring V_{full}, we obtain

$$V_{\text{frustum}} = V_{\text{full}} \left(1 - \frac{a^3}{A^3}\right) = V_{\text{full}} \cdot \frac{A^3 - a^3}{A^3}.$$

This completes the proof.

Theorem B (Volume in terms of the small cone)

Let $A \in \{R, H, L\}$ and $a \in \{r, h, \ell\}$ be corresponding linear dimensions of a right circular cone and a smaller cone removed by a plane parallel to the base. Then the volume of the truncated cone is

$$V_{\text{frustum}} = V_{\text{small}} \cdot \frac{A^3 - a^3}{a^3}.$$

Proof. As established in Theorem A, the similarity of the two cones implies that the ratio of their volumes equals the cube of the ratio of any corresponding linear dimensions. Thus,

$$\frac{V_{\text{small}}}{V_{\text{full}}} = \left(\frac{a}{A}\right)^3,$$

which can be rewritten as

$$V_{\text{full}} = V_{\text{small}} \cdot \frac{A^3}{a^3}.$$

The volume of the truncated cone is obtained by subtracting the volume of the smaller cone from that of the full cone:

$$V_{\text{frustum}} = V_{\text{full}} - V_{\text{small}} = V_{\text{small}} \cdot \frac{A^3}{a^3} - V_{\text{small}}.$$

Factoring V_{small}, we obtain

$$V_{\text{frustum}} = V_{\text{small}} \left(\frac{A^3}{a^3} - 1\right) = V_{\text{small}} \cdot \frac{A^3 - a^3}{a^3}.$$

This completes the proof.

2.3. Corollary Formulas

From Theorems A and B, the following particular cases follow immediately.

- **Radii:**

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{R^3 - r^3}{R^3}, \qquad V_{\text{frustum}} = V_{\text{small}} \cdot \frac{R^3 - r^3}{r^3}.$$

- **Heights:**

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{H^3 - h^3}{H^3}, \qquad V_{\text{frustum}} = V_{\text{small}} \cdot \frac{H^3 - h^3}{h^3}.$$

- **Slant heights:**

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{L^3 - \ell^3}{L^3}, \qquad V_{\text{frustum}} = V_{\text{small}} \cdot \frac{L^3 - \ell^3}{\ell^3}.$$

2.4. On the Coefficient $A^3 - a^3$

The general volume formulas obtained in Theorems A and B can be written using the common coefficient

$$k = A^3 - a^3,$$

where A and a are corresponding linear dimensions of the full and the removed cones.

Dividing the volume of the full cone by A^3 yields a quantity that depends only on the shape of the cone and not on its size. Multiplying this normalized volume by the coefficient k gives

$$\frac{V_{\text{full}}}{A^3} \cdot k = V_{\text{frustum}}.$$

Similarly, dividing the volume of the smaller cone by a^3 produces the same normalized quantity. Multiplying it by the same coefficient k again yields

$$\frac{V_{\text{small}}}{a^3} \cdot k = V_{\text{frustum}}.$$

Thus, regardless of whether the volume of the full cone or the smaller cone is used, multiplication by the coefficient $k = A^3 - a^3$ leads to the same volume of the truncated cone. This explains the structure of the two general formulas and the role of the coefficient.

3. Examples

The following examples illustrate how the derived formulas arise naturally in different geometric situations, depending on which parameters are known.

Example 1

Problem: The full cone has a volume of $V_{\text{full}} = 264$ units^3. A frustum is obtained by cutting the cone with a plane parallel to the base, such that the area of the top face is one fourth of the area of the base:

$$\frac{A_{\text{top}}}{A_{\text{base}}} = \frac{1}{4}.$$

Find the volume of the frustum.

Solution:

Since the cutting plane is parallel to the base, the removed cone is similar to the original cone. Thus, the ratio of the areas of the bases determines the ratio of the corresponding radii, and consequently the ratio of the corresponding linear dimensions of the two cones.

From the formula for the area of a circle:

$$\frac{\pi r^2}{\pi R^2} = \frac{1}{4} \quad \Rightarrow \quad \left(\frac{r}{R}\right)^2 = \frac{1}{4} \quad \Rightarrow \quad \frac{r}{R} = \frac{1}{2}.$$

Since the two cones are similar, the ratio of their volumes equals the cube of the ratio of their corresponding linear dimensions. Therefore, the volume of the removed cone is equal to $\left(\frac{r}{R}\right)^3$ of the volume of the full cone, and the volume of the frustum is obtained by subtracting this quantity from the volume of the full cone. This yields the corresponding volume relation

$$V_{\text{frustum}} = V_{\text{full}} \left(1 - \left(\frac{r}{R}\right)^3\right).$$

Substituting:

$$V_{\text{frustum}} = 264 \cdot \left(1 - \left(\frac{1}{2}\right)^3\right) = 264 \cdot \frac{7}{8} = \boxed{231 \text{ units}^3}.$$

Example 2

Problem: A frustum is formed from a cone with base radius $R = 6$, full height $H = 9$, and top cut height $h = 6$. Find the volume of the frustum.

Solution:

The full cone and the removed cone are similar, since the cutting plane is parallel to the base. Because the vertical heights of both cones are known, the ratio of their corresponding linear dimensions is determined by the ratio

$$\frac{h}{H} = \frac{6}{9}.$$

By geometric similarity, the ratio of the volumes of the two cones equals the cube of the ratio of their corresponding linear dimensions. Therefore, the volume of the removed cone is $\left(\frac{h}{H}\right)^3$ of the volume of the full cone, and the volume of the frustum is obtained by subtracting this quantity from the volume of the full cone. This leads to the expression

$$V_{\text{frustum}} = \frac{1}{3}\pi R^2 H \left(1 - \left(\frac{h}{H}\right)^3\right) = \frac{\pi R^2}{3H^2}(H^3 - h^3).$$

Substituting:

$$R^2 = 6^2 = 36, \quad H^3 - h^3 = 9^3 - 6^3 = 729 - 216 = 513,$$

$$V_{\text{frustum}} = \frac{\pi \cdot 36}{3 \cdot 81} \cdot 513 = 76\pi.$$

$$V \approx \boxed{238.76 \text{ units}^3}.$$

Example 3

Problem: A frustum is formed from a right circular cone with slant height $L = 10$. A parallel cut forms a smaller cone with slant height $\ell = 6$, whose volume is $V_{\text{small}} = 54\pi$. Find the volume of the frustum.

Solution:

In this case, the slant heights of the full cone and the removed cone are known, along with the volume of the smaller cone. Since the two cones are similar, the ratio of their volumes equals the cube of the ratio of their corresponding linear dimensions, which in this case are the slant heights.

Therefore, the volume of the frustum is obtained by subtracting the volume of the smaller cone from that of the full cone, leading to the relation

$$V_{\text{frustum}} = V_{\text{small}} \left(\frac{L^3 - \ell^3}{\ell^3} \right).$$

Substituting:

$$V_{\text{frustum}} = 54\pi \cdot \frac{10^3 - 6^3}{6^3} = 54\pi \cdot \frac{784}{216}.$$

$$V = \frac{42336\pi}{216} = \boxed{196\pi \approx 615.75 \text{ units}^3}.$$

These examples demonstrate that, depending on which dimensions are known — radii, vertical heights, or slant heights — the corresponding formula follows naturally from geometric similarity, and all resulting approaches lead to the same volume of the truncated cone.

4. Applications

The two theorems and their respective equivalent forms allow for the calculation of the volume of a truncated cone, utilizing different elements, thus reflecting the real-world occurrences in which there would be a limited amount of data. They present a unique method of solving problems using relationships between elements and are highly applicable in most practical and educational contexts in which the shape of a right circular frustum is involved, owing to the number of variations and the ability to approach different problems in a facile way.

Below are several real-world scenarios in which these theorems prove particularly useful, depending on which dimensions are available:

- **Packaging and Manufacturing (e.g., paper cups, plastic containers):** Products are typically designed with known base and top radii and a measured vertical height. Here, the formulas involving **radii and height** are most applicable:

$$V_{\text{frustum}} = \frac{\pi H}{3R}(R^3 - r^3) \quad \text{or} \quad V_{\text{frustum}} = \frac{1}{3}\pi R^2 H \cdot \frac{R^3 - r^3}{R^3}$$

- **Architecture and Construction (e.g., conical roofs, towers, trun cated domes):** Blueprints often include total vertical height and the height at which a roof is truncated. When the top part is removed, the volume can be calculated using the formulas involving **heights**:

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{H^3 - h^3}{H^3} \quad \text{or} \quad V_{\text{frustum}} = V_{\text{small}} \cdot \frac{H^3 - h^3}{h^3}$$

- **Metalwork and Casting (e.g., candle molds, bell shapes):** Molds often have measurable slant heights due to how they are cast or cut. In such cases, the formulas using **slant heights** are the most practical:

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{L^3 - \ell^3}{L^3} \quad \text{or} \quad V_{\text{frustum}} = V_{\text{small}} \cdot \frac{L^3 - \ell^3}{\ell^3}$$

- **Forestry and Agriculture (e.g., estimating tree trunk or grain bin volume):** When top and base diameters are known, along with the vertical length or cut height, the following formulas are ideal:

$$V_{\text{frustum}} = \frac{\pi h}{3r}(R^3 - r^3) \quad \text{or} \quad V_{\text{frustum}} = \frac{1}{3}\pi r^2 h \cdot \frac{R^3 - r^3}{r^3}$$

- **Chimneys and Furnaces (industrial design):** The inner and outer radii, as well as slant lengths (due to angled construction), are usually known. Slant-based formulas apply well:

$$V_{\text{frustum}} = \frac{1}{3}\pi r^2 h \cdot \frac{L^3 - \ell^3}{\ell^3} \quad \text{or} \quad V_{\text{frustum}} = \frac{1}{3}\pi R^2 h \cdot \frac{L^3 - \ell^3}{L^3}$$

- **Fluid Capacity Estimation (e.g., water tanks with conical sections):** If the full height and height of fluid level (cut-off level) are known, then **height-based** formulas like:

$$V_{\text{frustum}} = V_{\text{full}} \cdot \frac{H^3 - h^3}{H^3}$$

allow for efficient estimation of liquid volume.

In all cases, the different forms of the volume formula ensure that regardless of which elements (radii, vertical height, or slant height) are known, the correct volume can be calculated without the need to reconstruct missing dimensions. Such versatility highlights the educational and applied importance of the derived formulas, particularly in connecting geometric reasoning with real-world problem solving.

Other Applications

In addition to the core applied fields, truncated cones appear frequently in various areas of engineering, design, and daily life. Below is a categorized list of such applications:

- Engineering and Construction: Cooling towers; Funnels; Hoppers and Silos; Chimneys and Flues
- Manufacturing and Product Design: Cups and Glasses; Buckets; Pails; Machine Parts
- Aerospace and Marine: Rocket nozzles; Ship Hulls and Buoyancy devices
- Everyday Objects: Traffic cones; Lampshades; Speaker cones
- Mathematics and Education: Geometry and Volume Problems; Examples of Similarity; CAD Modelling

5. Conclusion

This study presents a comprehensive novel approach to calculating the volume of a truncated right circular cone, and illustrates the relationship between elements in a three-dimensional space. By utilizing two theorems with their six equivalent forms, based on radii, heights, or slanted heights, this method allows for a flexible application in search of every element of the conical structure, relying on the available data.

In addition to that, the results of the research become highly relevant in situations where said data is limited, for example in cases in which only the relationship between two dimensions and the volume of the cut-out cone are given. Thus, the formulas provide a reliable and facile way of solving problems involving frustums without the need of reconstructing the whole cone, which is of great worth in key industries such as manufacturing, architecture, and engineering.

Beyond the practical uses of this study, the theorems also offer theoretical value in Mathematics Education by illustrating in a clear manner the usefulness of geometric similarity and proportional reasoning, and serving as effective tools for teaching volume calculation and problem-solving strategies in diverse situations (Smith 2015; Posamentier & Salkind, 2017).

Finally, the novel method of analysis employed in this study not only succeeds in simplifying complex geometric problems, but also becomes a viable basis for further future research, such as exploring analogous theorems for surface area, adapting the same logic to other solids like pyramidal structures, or implementing interactive computational tools for broader use.

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