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Educational Issues
Въпроси на преподаването

PROBLEMS FOR PLANE FIGURES RELATED TO CONIC SECTIONS

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Abstract. This work is dedicated to problems involving plane figures defined by points related to conic sections. The presented problems involve finding elements or areas of convex polygons whose vertices lie on ellipses, hyperbolas, and parabolas. Extremal area problems are also considered.

Keywords: conic sections; ellipse; hyperbola; parabola; circle; polygons.

1. Introduction

In the elementary geometry education, students learn to solve various problems involving special positions of triangles and quadrilaterals relative to a circle without using a coordinate system. Students in Bulgarian high schools study the methods of analytic geometry in the mathematics profiled training where they also learn about conic sections (circle, ellipse, hyperbola and parabola). In the 11th grade, they are introduced to the equations of the four conics and their basic elements (center, vertices, foci, directrix of a parabola, asymptotes of a hyperbola, etc.), and in the 12th grade, they are taught the analytic expressions of tangent lines to second degree curves. A separate lesson in the 11th grade is dedicated to problems involving finding some elements of triangles with vertices given as points in the Cartesian coordinate system in the plane: midpoints, centroid, equations of medians and altitudes, side lengths and areas. Motivated by the above, in this article, we present problems for plane figures defined by elements of conic sections or points on the curves with the goal of extending students' understanding of the circle to other conic sections. Such problems with interrelated subject matters are designed to help the consolidation of students' knowledge simultaneously in several branches of mathematics: analytic geometry, planimetry, trigonometry, calculus, and algebra, and to demonstrate the applications of one field of mathematics to another. The presented problems are suitable for summary lessons and extracurricular activities for expanding and deepening the knowledge of analytic geometry and conic sections. Students are encouraged to use dynamic geometric software, such as GeoGebra, Desmos, etc., to visualize the objects considered in the problems.

2. Preliminaries

Throughout this work we will consider the equations of the conic sections in the Cartesian coordinate system Oxy. Ellipses and hyperbolas will be centered at the origin O with Ox being the focal axis, unless stated otherwise. If an ellipse is given by the equation

$$b^2x^2 + a^2y^2 = a^2b^2, \quad a > b > 0, \tag{1}$$

then its vertices are the points $(\pm a; 0)$ and $(0; \pm b)$, and its foci are $F_1(-c; 0)$ and $F_2(c; 0)$, where $c = \sqrt{a^2 - b^2}$. In the case $a = b$, equation (1) defines a circle with radius a. If a hyperbola is given by the equation

$$b^2x^2 - a^2y^2 = a^2b^2, \quad a, b > 0, \tag{2}$$

then its vertices are $(\pm a; 0)$, and its foci are $F_1(-c; 0)$ and $F_2(c; 0)$, where $c = \sqrt{a^2 + b^2}$. The asymptotes of the hyperbola (2) are the lines given by the equations $bx \pm ay = 0$. In the case $a = b$, the hyperbola (2) is called rectangular.

A parabola will have its vertex at O and will be defined by the equation

$$y^2 = 2px, \quad p \neq 0, \tag{3}$$

unless stated otherwise. Then, its focus is $F\left(\frac{p}{2}; 0\right)$ and its directrix is $g: x = -\frac{p}{2}$.

Let $M(x_0; y_0)$ be an arbitrary point on the conic section (1), (2) or (3). Then, the tangent t to the curve at point M is given by $t: b^2x_0x + a^2y_0y = a^2b^2$ in the case of an ellipse, by $t: b^2x_0x - a^2y_0y = a^2b^2$ in the case of a hyperbola, and by $t: y_0y = p(x + x_0)$ in the case of a parabola.

The dot product of vectors $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\vec{b} = |\vec{a}||\vec{b}| \cos \measuredangle(\vec{a}, \vec{b})$. Then, $\vec{a}^2 = \vec{a}\vec{a} = |\vec{a}|^2$. The non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\vec{b} = 0$.

In some of the problems, it will be more convenient to use a formula for the area of a triangle (or parallelogram) which may be unfamiliar to school students in its vector form. Let $\vec{a}$ and $\vec{b}$ be non-collinear vectors. Then, the area S of the triangle defined by $\vec{a}$ and $\vec{b}$ is calculated using the formula

$$S = \frac{1}{2}\sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}\vec{b})^2}. \tag{4}$$

Formula (4) is derived from the familiar expression $S = \frac{|\vec{a}||\vec{b}|\sin\measuredangle(\vec{a},\vec{b})}{2}$ by squaring both sides.

3. Problems for finding elements and areas of plane figures

Problem 1. Find the side of an equilateral triangle inscribed in: a) an ellipse; b) a branch of a hyperbola; c) a parabola, in such a way that one of the triangle's vertices coincides with a vertex of the conic section.

Solution. a) Let the ellipse be given by (1), and the common vertex be $A(a;0)$. Then, if the other two vertices of the triangle are denoted by B and C, due to the symmetry of the ellipse, BC is perpendicular to the Ox axis. Hence, $B(x_0;-y_0)$ and $C(x_0;y_0)$, $0 \leq x_0 < a$, $y_0 > 0$ (Fig. 1). Then, $\overrightarrow{BC}(0;2y_0)$, $\overrightarrow{BA}(a-x_0;y_0)$. From the condition $|BC| = |AB| = |AC|$, we obtain $a - x_0 = y_0\sqrt{3}$ which together with (1) yields $|BC| = 2y_0 = \frac{4\sqrt{3}ab^2}{a^2+3b^2}$. If one of the triangle vertices coincides with a vertex on the minor axis of the ellipse, then by similar reasoning we find that the length of the triangle's side is $\frac{4\sqrt{3}a^2b}{3a^2+b^2}$.

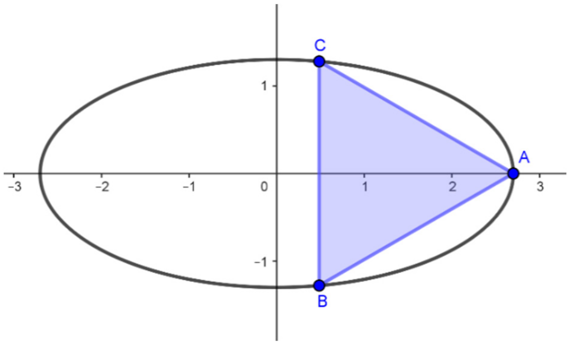

Figure 1. Drawing for Problem 1 a)

b) Analogously, for the hyperbola given by (2), we obtain that the triangle's side is $\frac{4\sqrt{3}ab^2}{a^2-3b^2}$ if $a > b\sqrt{3}$, and in the case of a parabola given by (3), in c) we get $4\sqrt{3}p$, $p > 0$.

Problem 2. Consider the parabola $y^2 = 2px$, $p > 0$. A circle centered at the vertex of the parabola intersects the curve at two points so that the triangle defined by these points and the parabola's vertex is equilateral. Find the circle's radius and area (Fig. 2).

Solution. Let A and B be the points of intersection. They are symmetrical with respect to the Ox axis. We can denote their coordinates by $A\left(\frac{y_0^2}{2p}; -y_0\right)$ and $B\left(\frac{y_0^2}{2p}; y_0\right)$, $y_0 > 0$. Since ΔOAB is equilateral, by Problem 1 we have $y_0 = 2p\sqrt{3}$. The circle's equation is $x^2 + y^2 = r^2$, $r > 0$. The condition that point A lies on the circle yields $r = 4p\sqrt{3}$, and then the circle's area is $S = 48\pi p^2$.

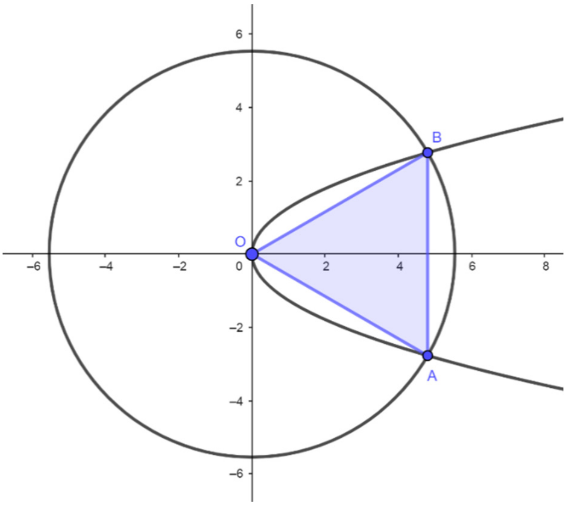

Figure 2. Drawing for Problem 2

Problem 3. Find the area of an isosceles ΔABC ($|AB| = |AC|$, $\measuredangle BAC = \alpha$) inscribed in: a) a branch of a rectangular hyperbola; b) a parabola, in a way that A is a vertex of the conic section, and B and C are points on the curve (Fig. 3).

Solution. a) Let the hyperbola be given by $x^2 - y^2 = a^2$ $(a > 0)$, and the triangle's vertices be $A(a; 0)$, $B(x_0; -y_0)$, $C(x_0; y_0)$, $x_0 > a$, $y_0 > 0$. Then, by the dot product formula for $\overrightarrow{AB}(x_0 - a; -y_0)$ and $\overrightarrow{AC}(x_0 - a; y_0)$ and the curve's equations, we obtain $\overrightarrow{AB}.\overrightarrow{AC} = 2a(a - x_0)$, $|\overrightarrow{AB}| = |\overrightarrow{AC}| = \sqrt{2x_0(x_0 - a)}$ and compute $\cos\alpha = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = -\frac{a}{x_0}$. We get $x_0 = -\frac{a}{\cos\alpha}$, $\alpha \neq \frac{\pi}{2}$, and $y_0 = a|\tan\alpha|$. Then, using (4) we obtain $S_{ABC} = y_0(x_0 - a)$ and thus

$$S_{ABC} = \frac{4a^2\tan\frac{\alpha}{2}}{\left(1-\tan^2\frac{\alpha}{2}\right)^2}.$$

b) In a similar manner to a), let the parabola be given by (3), and the triangle's vertices be $A \equiv O(0; 0)$, $B\left(\frac{y_0^2}{2p}; -y_0\right)$, $C\left(\frac{y_0^2}{2p}; y_0\right)$, $y_0, p > 0$. By calculating $\cos\alpha$, we obtain $y_0 = 2p\cot\frac{\alpha}{2}$. Then, $S_{ABC} = 4p^2\cot^3\frac{\alpha}{2}$.

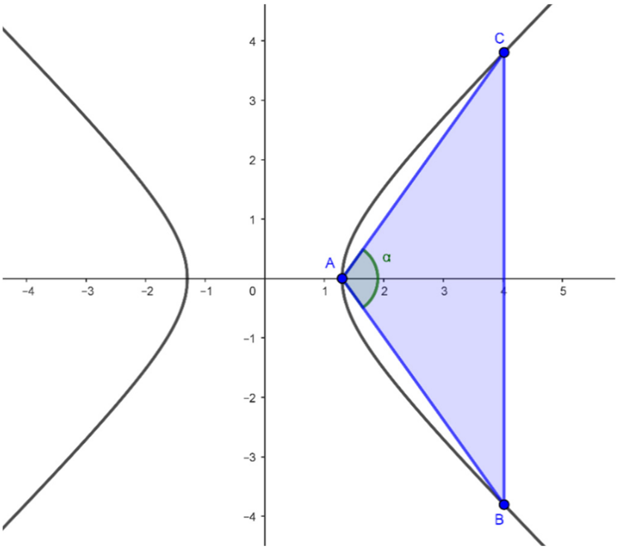

Figure 3. Drawing for Problem 3 a)

Problem 4. (Atanasyan et al., 1964) Find the sides of an isosceles triangle inscribed in a parabola in such a manner that one of its vertices coincides with the parabola's vertex, and its orthocenter is the parabola's focus (Fig. 4).

Solution. Let us use the notations in Problem 3 b). Since $\overrightarrow{FC} \perp \overrightarrow{OB}$, we have $\overrightarrow{FC}.\overrightarrow{OB} = 0$, where F is the focus of the parabola. Thus, we obtain $y_0 = p\sqrt{5}$ and the sides of the triangle have lengths $|OB| = |OC| = \frac{3}{2}p\sqrt{5}$, $|BC| = 2p\sqrt{5}$.

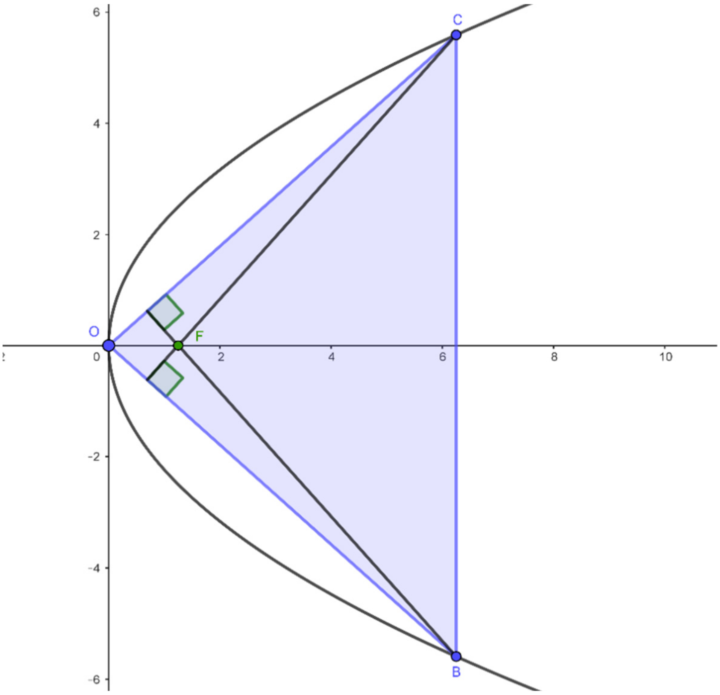

Figure 4. Drawing for Problem 4

Problem 5. (Atanasyan et al., 1964) Find the area of a square circumscribed around an ellipse (Fig. 5).

Solution. Due to the symmetry of the ellipse, the vertices of the square should lie on the axes and also be placed symmetrically. Hence, the vertices are $(\pm x_0; 0)$ and $(0; \pm x_0)$, $x_0 > 0$. Let $A(x_0; 0)$ and $B(0; x_0)$ be two of the square's vertices. Then, the line $AB: x + y - x_0 = 0$ should be tangent to the ellipse (1), and hence the system of equations of the ellipse and the line AB should have one solution. Thus, the discriminant of the quadratic equation

$$(a^2+b^2)x^2-2a^2x_0x+a^2(x_0^2-b^2)=0$$

should be equal to zero. From this condition, we obtain $x_0=\sqrt{a^2+b^2}$ from where $S=2(a^2+b^2)$. In (Rangelova et al., 2025), it is proved that the loci of the points from which the ellipse (1) is seen at a right angle is the circle given by the equation $x^2+y^2=a^2+b^2$. The vertices of the square are the circle's intersection points with the axes (Fig. 5).

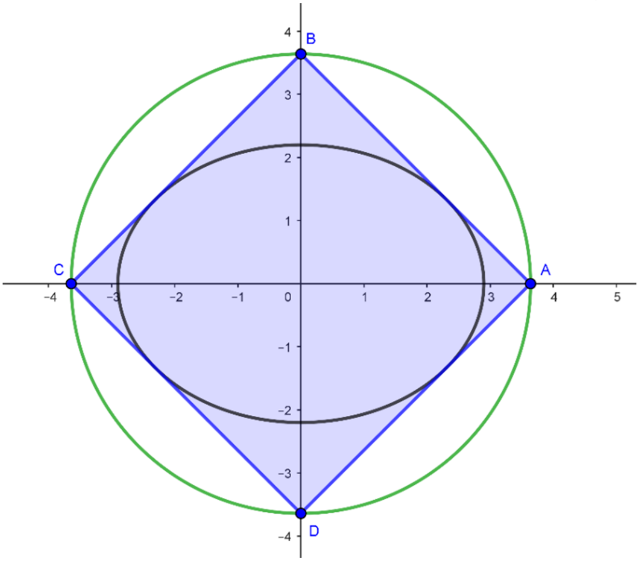

Figure 5. Drawing for Problem 5

Problem 6. (Paskalev, 1985; Stanilov & Borisov, 1988) Consider the triangle defined by the asymptotes of a hyperbola and an arbitrary tangent line to the curve. Prove that: a) the centroid of the triangle lies on the line through the center of the hyperbola and the tangent point; b) the triangle has constant area (Fig. 6).

Solution. a) Let the hyperbola be given by (2). Then, its asymptotes are the lines with the equations $l_{1,2}\colon bx\pm ay=0$. Let $M(x_0;y_0)$ be an arbitrary point on the hyperbola. The tangent to the hyperbola at M is $t\colon b^2x_0x-$

$a^2y_0y = a^2b^2$. By finding the intersection points of $l_{1,2}$ and t we obtain the vertices of ΔOAB

$$A\left(\frac{a^2b}{bx_0-ay_0};\frac{ab^2}{bx_0-ay_0}\right),\ B\left(\frac{a^2b}{bx_0+ay_0};-\frac{ab^2}{bx_0+ay_0}\right).$$

Then, we prove that $\overrightarrow{OA}+\overrightarrow{OB}=2\overrightarrow{OM}$, i.e. the tangent point M is the midpoint of AB and hence OM is a median of the triangle.

b) To find the area S_{OAB} we use formula (4). Having in mind that the coordinates of point $M(x_0;y_0)$ satisfy the equation of the hyperbola (2), we compute $\overrightarrow{OA}.\overrightarrow{OB}=a^2-b^2$, and $\overrightarrow{OA}^2.\overrightarrow{OB}^2=(a^2+b^2)^2$. Then, by (4) we get $S_{OAB}=ab$ which does not depend on the point $M(x_0;y_0)$, and hence S_{OAB} is constant.

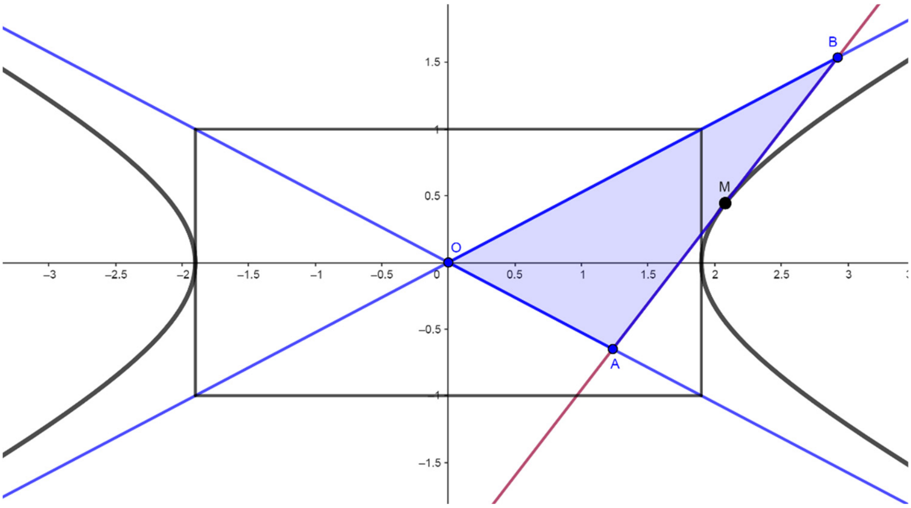

Figure 6. Drawing for Problem 6

Students are familiar with the function $y=\frac{1}{x}$ and its graphical representation. Since the hyperbola $xy=1$ is obtained from the hyperbola $x^2-y^2=2$ by a 45°-rotation, Problem 6 b) can be formulated also in the following way

Problem 7. Find the area of the triangle bounded by a tangent to the hyperbola $xy=a$ $(a\neq 0)$ and the coordinate axes.

Solution. The axes of symmetry of the hyperbola with equation $xy = a$ are the coordinate bisectors $y = \pm x$ and Ox and Oy are its asymptotes (fig. 7).

The line $t: y = kx + n$ through an arbitrary point $M\left(x_0; \frac{a}{x_0}\right)$, $x_0 \neq 0$, on the hyperbola is tangent to the curve if and only if the system of the line and the curve's equation has only one solution. Thus, we get $k = -\frac{n^2}{4a}$. Then, considering that the line t passes through M, we obtain $n = \frac{2a}{x_0}$ and the equation

$t: ax + x_0^2 y - 2ax_0 = 0$. The intersection points of t with Ox and Oy are $A(2x_0; 0)$ and $B\left(0; \frac{2a}{x_0}\right)$, respectively. Then, we calculate $S_{OAB} = 2|a| =$ const.

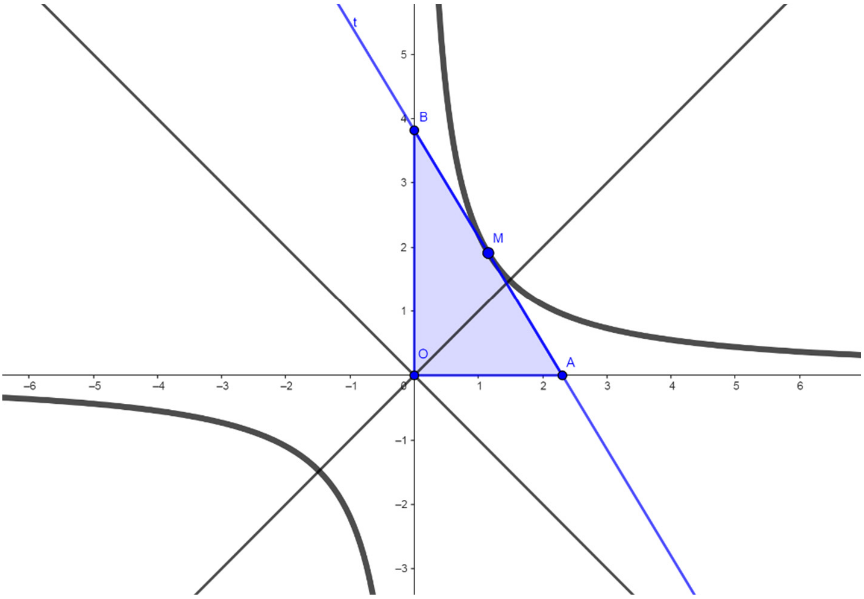

Figure 7. Drawing for Problem 7

Problem 8. The vertices of a right ΔABC ($\measuredangle BAC = \frac{\pi}{2}$) lie on the hyperbola $xy = 1$ in such a way that A is a vertex of the curve and $|AB|:|AC| = \lambda:\mu$. Find the area of ΔABC (Fig. 8).

Solution. The vertices of the hyperbola are $(1;1)$ and $(-1;-1)$ lying on its real axis $x-y=0$. Let the triangle vertices be $A(1;1)$, $B\left(x_1;\frac{1}{x_1}\right)$ and $C\left(x_2;\frac{1}{x_2}\right)$, $x_1 \neq x_2$, $x_1, x_2 \neq 0, \pm 1$. Then, since $\overrightarrow{AB} \perp \overrightarrow{AC}$ we have $\overrightarrow{AB}.\overrightarrow{AC}=0$ which yields $x_1x_2=-1$. Next, by the condition $\mu^2\overrightarrow{AB}^2=\lambda^2\overrightarrow{AC}^2$ we arrive at the equation

$$(\lambda^2-\mu^2)\left(x_1^2+\frac{1}{x_1^2}\right)+2(\lambda^2+\mu^2)\left(x_1+\frac{1}{x_1}\right)+2(\lambda^2-\mu^2)=0.$$

Let us set $u=x_1+\frac{1}{x_1}$. Then, we get $(\lambda^2-\mu^2)u^2+2(\lambda^2+\mu^2)u=0$. In the case $\lambda \neq \mu$, we obtain the solution $u=-\frac{2(\lambda^2+\mu^2)}{\lambda^2-\mu^2}$. Since $S=\frac{\mu}{2\lambda}|AB|^2$ we calculate $\overrightarrow{AB}^2=u^2-2u=\frac{8\lambda^2(\lambda^2+\mu^2)}{(\lambda^2-\mu^2)^2}$ and finally we obtain $S=\frac{4\lambda\mu(\lambda^2+\mu^2)}{(\lambda^2-\mu^2)^2}$.

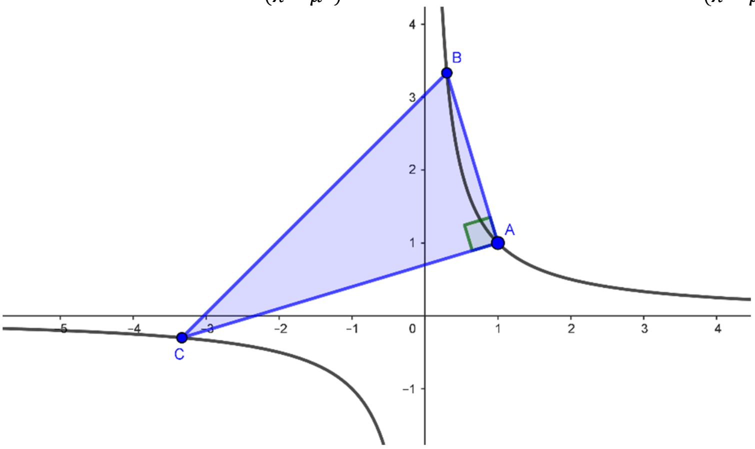

Figure 8. Drawing for Problem 8

Problem 9. Consider the triangle formed by three tangent lines to a parabola. Prove that: a) (Paskalev, 1979) its orthocenter lies on the directrix of the parabola; b) (Paskalev, 1985) the circumscribed circle passes through the focus of the parabola.

Solution. a) Let us consider the tangent lines t_1 and t_2 to the parabola (3) at two arbitrary points $M_1(x_1;y_1)$ and $M_2(x_2;y_2)$, $y_1 \neq y_2$, respectively, which are defined by $t_i\colon px-y_iy+px_i=0$ $(i=1,2)$. Considering that $x_i=\frac{y_i^2}{2p}$, the lines t_1 and t_2 meet at point $A\left(\frac{y_1y_2}{2p};\frac{y_1+y_2}{2}\right)$. If $M_3(x_3;y_3)$ is a third point on the parabola, the tangent to the parabola at M_3 is given by $t_3\colon px - y_3y + px_3 = 0$. The altitude through A lies on the line h_A with the equation

$$h_A\colon y_3\left(x-\frac{y_1y_2}{2p}\right)+p\left(y-\frac{y_1+y_2}{2}\right)=0.$$

The intersection point of the line h_A and the parabola's directrix $g\colon x=-\frac{p}{2}$ is $H\left(-\frac{p}{2};\frac{y_1+y_2+y_3}{2}+\frac{y_1y_2y_3}{2p^2}\right)$.

Since the second coordinate of H is a symmetrical expression with respect to y_1, y_2 and y_3, H is the intersection point of each altitude of the triangle with the directrix. Hence, H lies on each of the three altitudes and is the triangle's orthocenter (Fig. 9).

b) One way to find the circumcenter O' is to use Euler's line theorem which states that O', the orthocenter H and the centroid G of a triangle are collinear and $\overrightarrow{O'H}=3\overrightarrow{O'G}$. Thus, for the coordinates of G and O' we obtain

$$G\left(\frac{y_1y_2+y_1y_3+y_2y_3}{6p};\frac{y_1+y_2+y_3}{3}\right),\quad O'\left(\frac{y_1y_2+y_1y_3+y_2y_3}{4p}+\frac{p}{4};\ \frac{y_1+y_2+y_3}{4}-\frac{y_1y_2y_3}{4p^2}\right).$$

Then, we prove that $\left|\overrightarrow{FO'}\right|=\left|\overrightarrow{AO'}\right|$ which yields that the focus F lies on the circumscribed circle.

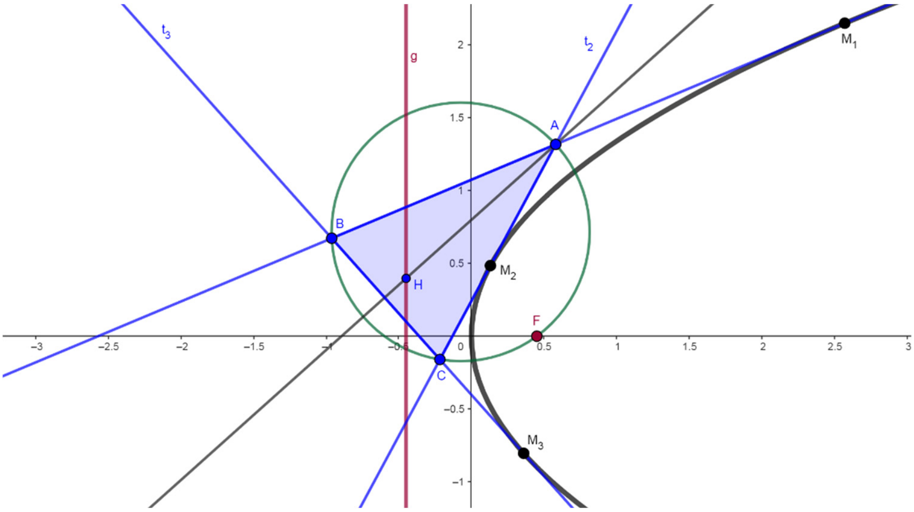

Figure 9. Drawing for Problem 9

4. Extremal area problems

In the 12th grade of the math profiled training, students learn to solve geometric extremal value problems in the plane and in 3D space. Here we present such problems for areas of convex polygons related to conic sections. To solve some of the problems, we use the means of mathematical analysis, and others we solve with the help of the AM-GM inequality $x_1 + x_2 \geq 2\sqrt{x_1 x_2}$ for all $x_1, x_2 \geq 0$ (equality is reached when $x_1 = x_2$).

Problem 10. Find the maximal area of a rectangle inscribed in an ellipse.
Solution. Let the ellipse be defined by (1). The rectangle's vertices are the points $(\pm x_0; \pm y_0)$, $x_0, y_0 > 0$. From (1) we obtain $y_0 = \frac{b}{a}\sqrt{a^2 - x_0^2}$. Then, the area of the rectangle is $S = 4x_0 y_0 = 4\frac{b}{a}x_0\sqrt{a^2 - x_0^2}$. Let us consider the function $f(x) = x\sqrt{a^2 - x^2}$ for $0 < x < a$. The first derivative $f'(x) = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$ vanishes for $x = \frac{a\sqrt{2}}{2}$, and changes sign from plus to minus at this point. Hence, we have $f_{\max} = f\left(\frac{a\sqrt{2}}{2}\right)$. Thus, the maximal value of the area is $S_{\max} = 2ab$, and it is the area of the inscribed rectangle with vertices $\left(\pm\frac{a\sqrt{2}}{2}; \pm\frac{b\sqrt{2}}{2}\right)$ (Fig. 10).

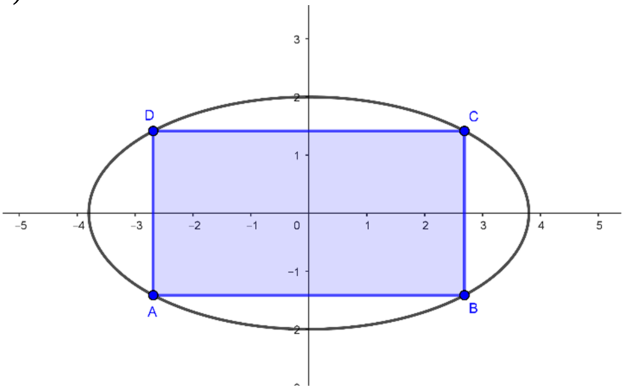

Figure 10. Drawing for Problem 10

Problem 11. (Stanilov & Borisov, 1988) Find a point on an ellipse such that the triangle formed by the tangent line at this point and the axes has a minimal area.

Solution. Due to the symmetry of the ellipse (1), we can consider only points in the first quadrant. The tangent $t\colon b^2x_0x+a^2y_0y=a^2b^2$ to the ellipse at an arbitrary point $M(x_0;y_0)$, $x_0,y_0>0$ intersects the axes at points $A\left(\frac{a^2}{x_0};0\right)$ and $B\left(0;\frac{b^2}{y_0}\right)$, respectively. Then, the area $S_{OAB}=\frac{a^2b^2}{2x_0y_0}$ is minimal when the product x_0y_0 is maximal. Having in mind that $y_0=\frac{b}{a}\sqrt{a^2-x_0^2}$ and the results in Problem 10, we obtain $S_{\min}=ab$ which is reached for the points $\left(\pm\frac{a\sqrt{2}}{2};\pm\frac{b\sqrt{2}}{2}\right)$ (Fig. 11).

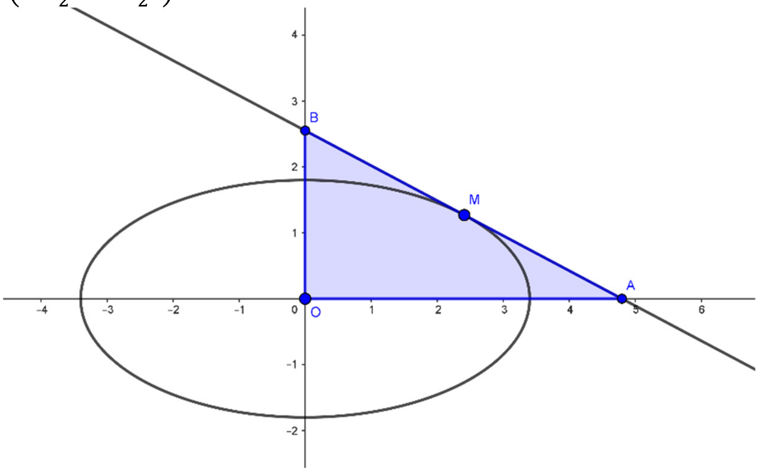

Figure 11. Drawing for Problem 11

Problem 12. Find the minimal area of a triangle formed by the asymptotes of a hyperbola and a line passing through its focus (Fig. 12).

Solution. Let $A\left(x_1;-\frac{b}{a}x_1\right)$ and $B\left(x_2;\frac{b}{a}x_2\right)$ be arbitrary points on the asymptotes $bx\pm ay=0$, respectively, of the hyperbola (2) and consider the focus $F(c;0)$. Then, the line AB passes through F if and only if $\overrightarrow{FA}$ and $\overrightarrow{FB}$ are collinear which is equivalent to c being the harmonic mean of x_1 and x_2, i.e. $c=\frac{2x_1x_2}{x_1+x_2}$. Then, by formula (4) we compute $S_{OAB}^2=\frac{b^2}{a^2}x_1^2x_2^2=\frac{(a^2+b^2)b^2}{a^2}.\frac{x_1^4}{(2x_1-c)^2}$. Let us consider the function $f(x)=\frac{x^4}{(2x-c)^2}$, $x>a$, $x\neq\frac{c}{2}$. The first derivative is $f'(x)=\frac{4x^3(x-c)}{(2x-c)^3}$. The function reaches its minimal value for the considered values of x at $x = c$, i.e. when ΔOAB is isosceles. Then, $S_{\min} = \frac{b}{a}(a^2 + b^2)$.

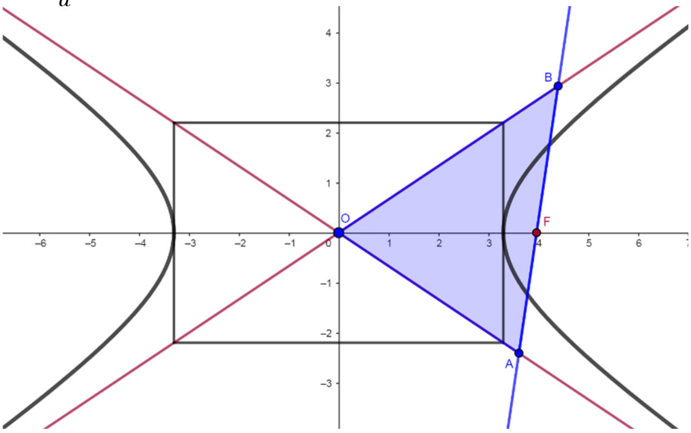

Figure 12. Drawing for Problem 12

Problem 13. Find the maximal area of a rectangle inscribed under the parabola given by the equations $y = a^2 - x^2$ $(a \neq 0)$ in such a way that two of its vertices lie on the Ox axis, and the other two lie on the curve (Fig. 13).

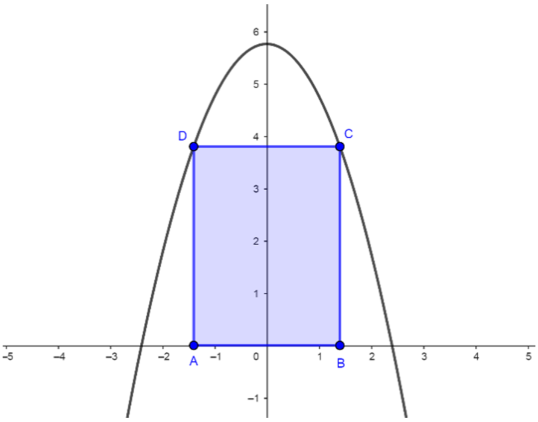

Figure 13. Drawing for Problem 13

Solution. Let the rectangle be $ABCD$ with A, B lying on Ox, and C, D lying on the parabola. Due to the parabola's symmetry around the Ox axis, the vertices are $A(-x_0; 0)$, $B(x_0; 0)$, $C(x_0; a^2 - x_0^2)$ and $D(-x_0; a^2 - x_0^2)$, $0 < x_0 < a$. Then, $S_{ABCD} = 2x_0(a^2 - x_0^2)$. Consider the function $f(x) = x(a^2 - x^2)$. The first derivative is $f'(x) = a^2 - 3x^2$. Then, $f(x)$ has one critical point for $0 < x < a$ which is $x = \frac{a\sqrt{3}}{3}$ and is a maximum. Thus, $S_{\max} = \frac{4a^3\sqrt{3}}{9}$.

Problem 14. Find the minimal area of a triangle inscribed in a parabola if one of its vertices coincides with the vertex of the parabola and the opposite side passes through the focus of the parabola (Fig. 14).

Solution. Let ΔOAB be inscribed in the parabola (3) in such a way that O is the parabola's vertex, and $A\left(\frac{y_1^2}{2p}; y_1\right)$, $B\left(\frac{y_2^2}{2p}; y_2\right)$, $y_1 \neq y_2$ lie on the parabola. Then, by formula (4) we calculate the area $S_{OAB}^2 = \frac{y_1^2 y_2^2 (y_1 - y_2)^2}{16p^2}$. The condition for the line AB to pass through the focus $F\left(\frac{p}{2}; 0\right)$, i.e. $\overrightarrow{FA}$ and $\overrightarrow{FB}$ to be collinear, yields $y_1 y_2 = -p^2$. Substituting this condition in the expression for the area we obtain $S_{OAB}^2 = \frac{p^2}{16}\left(y_1^2 + \frac{p^4}{y_1^2} + 2p^2\right)$. Then, by applying the AM-GM inequality we get $y_1^2 + \frac{p^4}{y_1^2} \geq 2p^2$, and hence $S_{OAB} \geq \frac{p^2}{2}$. The minimal area is reached in the case when $y_1 = p$, $y_2 = -p$, or vice-versa, i.e. when AB is perpendicular to Ox, and ΔOAB is isosceles.

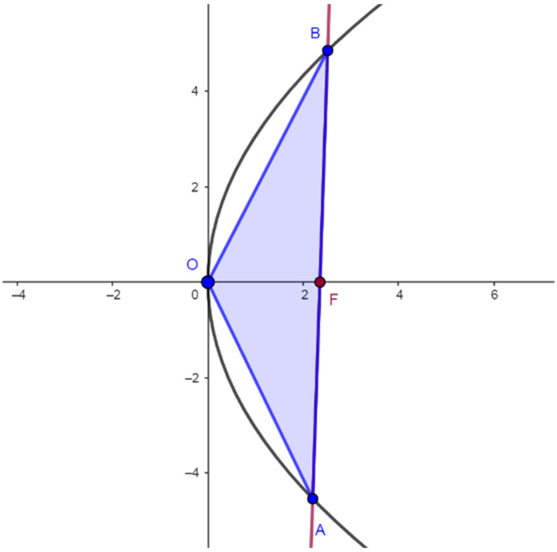

Figure 14. Drawing for Problem 14

Problem 15. Find the minimal area of a right triangle inscribed in a parabola if the right angle is at the parabola's vertex.

Solution. Let the vertices of the triangle be O, A and B. The condition for $\overrightarrow{OA}$ and $\overrightarrow{OB}$ to be perpendicular, i.e. $\overrightarrow{OA}.\overrightarrow{OB} = 0$, yields $y_1 y_2 = -4p^2$. Similarly, to Problem 14, we obtain $S_{OAB} \geq 4p^2$, and the minimum area is reached for the isosceles triangle with vertices $(0;0)$ and $(2p;\pm 2p)$ (Fig. 15).

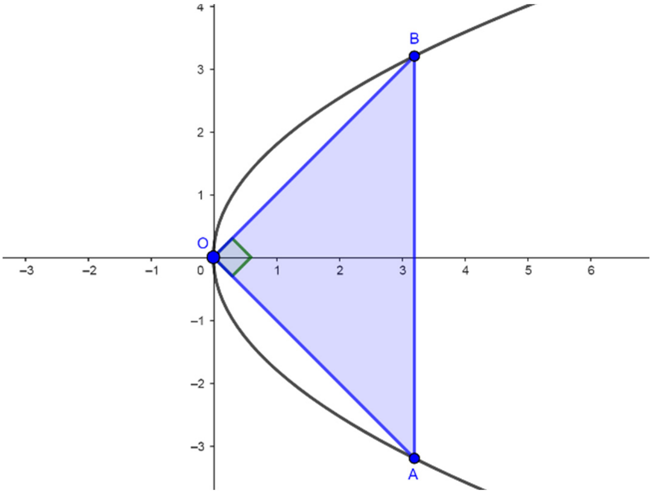

Figure 15. Drawing for Problem 15

Problem 16. Find the minimal area of a triangle bounded by two tangent lines to the parabola which intersect at a right angle.

Solution. Following the notations in Problem 14, the tangents to the parabola $t_i\colon px - y_i y + \frac{y_i^2}{2} = 0$ $(i = 1, 2)$ at points A and B, respectively, are perpendicular if and only if $y_1 y_2 = -p^2$. In this case the tangents meet at point $C\left(-\frac{p}{2}; \frac{y_1 + y_2}{2}\right)$ which lies on the directrix for any two tangent lines. One can easily prove that in this case the vectors $\overrightarrow{FA}$ and $\overrightarrow{FB}$ are collinear which means that the line AB passes through the focus F. To compute the area S_{ABC} we use formula (4), keeping in mind that $\overrightarrow{CA}.\overrightarrow{CB} = 0$ and substituting $y_2 = -\frac{p^2}{y_1}$. Thus, we obtain

$$4S_{ABC}^2 = \overrightarrow{CA}^2 \overrightarrow{CB}^2 = \left(\frac{p^2}{2} + \frac{1}{4}\left(y_1^2 + \frac{p^4}{y_1^2}\right)\right)^2 + \frac{1}{16}\left(2p^2 + y_1^2 + \frac{p^4}{y_1^2}\right)^2$$

$$+\frac{1}{4}\left(2p^2+y_1^2+\frac{p^4}{y_1^2}\right)\left(\frac{p^2}{2}+\frac{1}{4p^2}\left(y_1^4+\frac{p^8}{y_1^4}\right)+\frac{1}{2}\left(y_1^2+\frac{p^4}{y_1^2}\right)\right).$$

Then, using the AM-GM inequality we obtain $S_{ABC} \geq p^2$ with the minimal area being reached like in Problem 14 when $y_1 = p$, $y_2 = -p$, or vice-versa, i.e. when ΔABC is isosceles (Fig. 16).

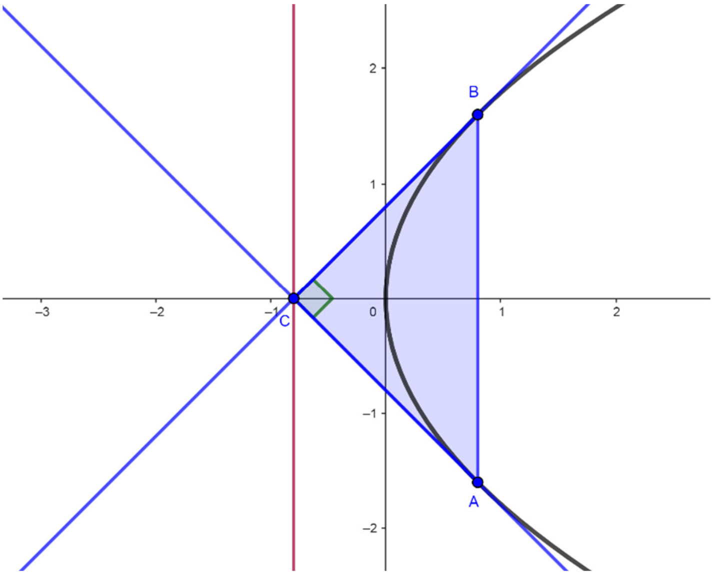

Figure 16. Drawing for Problem 16

5. Problems for students' self-training

Problem 17. Find the equation of:

a) an ellipse, if its minor axis is seen from its foci at an angle of $60°$ and the area of the quadrilateral formed by its vertices is S;

b) a hyperbola, if the triangle formed by the asymptotes and the line through a focus perpendicular to Ox is equilateral with area S;

c) a parabola, if inside it a trapezoid $ABCD$ ($AB \parallel CD$) with area S is inscribed so that CD passes through the focus and is perpendicular to Ox, and $AB = 2CD$.

Answer. a) $x^2 + 4y^2 = S$; b) $4x^2 - 12y^2 = 3\sqrt{3}S$; c) $3y^2 = 2\sqrt{2S}x$.

Problem 18. Find the area of the quadrilateral whose vertices are the endpoints of the chords through the foci perpendicular to the focal axis in the case of: a) an ellipse; b) a hyperbola.
Answer. a) $S = 4\frac{b^2}{a^2}\sqrt{a^2 - b^2}$; b) $S = 4\frac{b^2}{a^2}\sqrt{a^2 + b^2}$.

Problem 19. (Atanasyan et al., 1964) Find the area of a square whose vertices lie on the two branches of a hyperbola.
Short solution. The vertices of the square are $(\pm x_0; \pm x_0)$ and satisfy the hyperbola equation (2). Thus, $x_0 = \frac{ab}{\sqrt{b^2-a^2}}$ if $b > a$. Then, $S = \frac{4a^2b^2}{b^2-a^2}$.

Problem 20. Find the area of ΔOFH where F is a focus of a hyperbola and H is the foot of the perpendicular through F to an asymptote.
Answer. $S = \frac{ab}{2}$.

Problem 21. (Paskalev, 1985) Consider an arbitrary point M on a hyperbola and two lines through M which are parallel to its asymptotes. Prove that the parallelogram determined by these two lines and the asymptotes has constant area.
Short solution. Following the notations in Problem 6, the lines through $M(x_0; y_0)$ parallel to the asymptotes are given by $g_1\colon bx + ay - bx_0 - ay_0 = 0$ and $g_2\colon bx - ay - bx_0 + ay_0 = 0$. They intersect the asymptotes at points

$$A\left(\frac{bx_0+ay_0}{2b}; \frac{bx_0+ay_0}{2a}\right),\ B\left(\frac{bx_0-ay_0}{2b}; -\frac{bx_0-ay_0}{2a}\right),$$

respectively. Then, by formula (4) we obtain $S = \frac{ab}{2}$.

Problem 22. Find the area of an isosceles triangle inscribed in a parabola in such a way that one of its vertices coincides with the parabola's vertex, and its centroid is the parabola's focus. *Answer.* $S = \frac{3\sqrt{6}}{8}p^2$.

Problem 23. Find the area of the triangle bounded by a tangent to the parabola and the coordinate axes if the intersection point of the tangent and Ox lies on the directrix. *Answer.* $S = \frac{p^2}{8}$.

Problem 24. Find the ellipse for which the quadrilateral formed by its vertices and foci has the largest possible area.

Answer. The quadrilateral is a rhombus and its area is maximal when it becomes a square, i.e. when $b = c$. Hence, the answer is an ellipse with $a{:}b = \sqrt{2}{:}1$.

6. Conclusion

In this article, we have presented various problems about plane figures formed by points on conic sections, including special types of triangles (equilateral, isosceles and right triangles) and special quadrilaterals (parallelograms, rectangles and squares), in different positions relative to ellipses, hyperbolas and parabolas. The problems are solved by the methods of analytic geometry combined with knowledge in elementary geometry, calculus and algebra. In our opinion, the study of such problems contributes not only to the development of skills for applying the coordinate-vector approach in the plane but also facilitates the development of essential learning competencies like critical thinking, logical reasoning, and mathematical creativity.

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